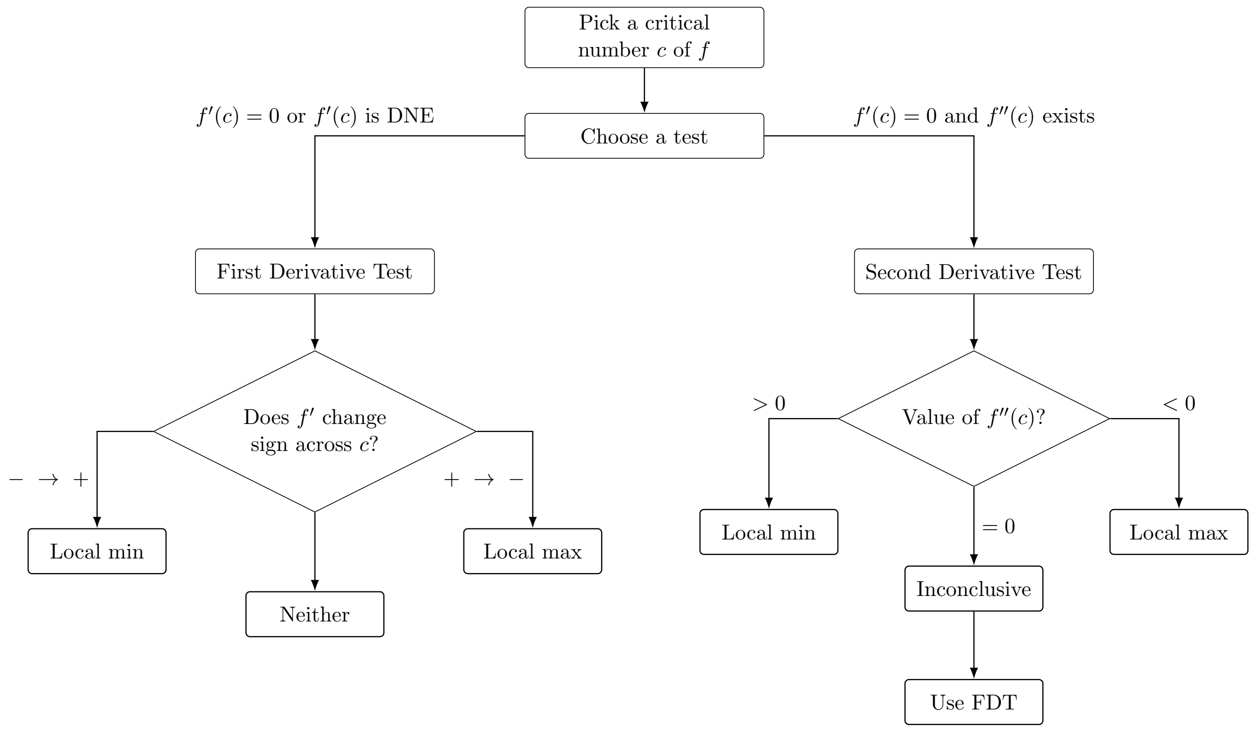
4.3 / 4.5 Curve Sketching

The main goal of this section is to decide what happens at a function's *critical numbers*—points in the domain where $f'(x) = 0$ or $f'(x)$ does not exist. For each such point, we want to determine whether f has a local maximum, a local minimum, or neither.



Combining this information with what we already know about increasing/decreasing and concavity will allow us to accurately analyze and sketch curves.

Theorem (Increasing/Decreasing Test)**.**

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof.

1. **Fix points.** Pick any x_1, x_2 in I with $x_1 < x_2$. According to the definition of "increasing," what do we need to prove?
$$f(\qquad) < f(\qquad)$$

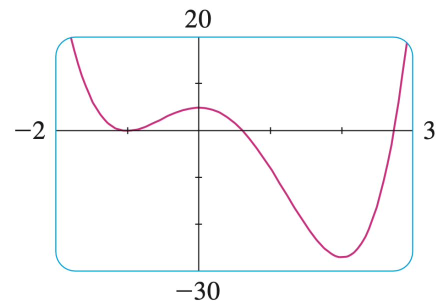
2. **Prepare to use MVT.** Because $f'(x) > 0$ on I, the function f is differentiable on (x_1, x_2) and hence continuous on $[x_1, x_2]$. By the MVT, there is some c in (x_1, x_2) so that
$$f(x_2) - f(x_1) =$$

3. **Sign analysis of the right-hand side.** Using the hypotheses $f'(x) > 0$ on I and $x_2 > x_1$, fill in the inequalities:
$$f'(\,c\,) \qquad 0 \qquad \text{and} \qquad x_2 - x_1 \qquad 0.$$
Therefore the product $f'(c)\big(x_2 - x_1\big)$ is ______________________________.

4. **Translate back to $f(x_2) - f(x_1)$.** We conclude that
$$f(x_2) - f(x_1) \qquad 0.$$
as well, so
$$f(\qquad) < f(\qquad)$$
We conclude that ____.

□

Example. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.



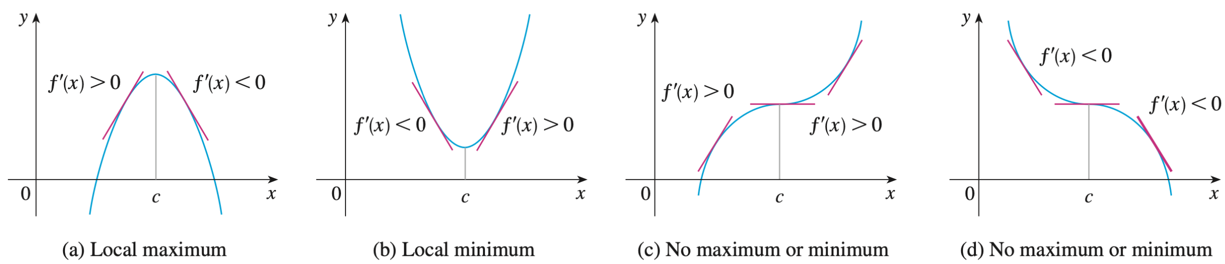
Theorem (First Derivative Test)**.** Let c be a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

Example. Visualizing the first derivative test.



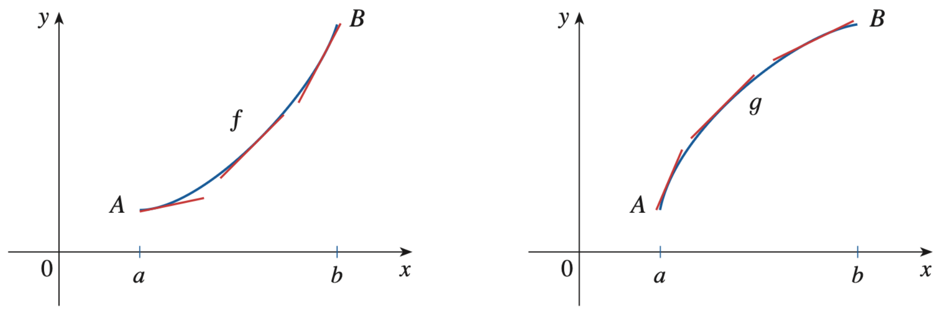
Example. Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Example. Find the local maximum and minimum values of the function $g(x) = x + 2\sin x$ on the interval $[0, 2\pi]$

Definition (Concavity)**.** Let I be an interval and $f(x)$ be differentiable on I.

(a) f is concave upward on I if, for every x in I, the graph of f lies above its tangent line at x.

(b) f is concave downward on I if, for every x in I, the graph of f lies below its tangent line at x.



(a) Concave upward (b) Concave downward

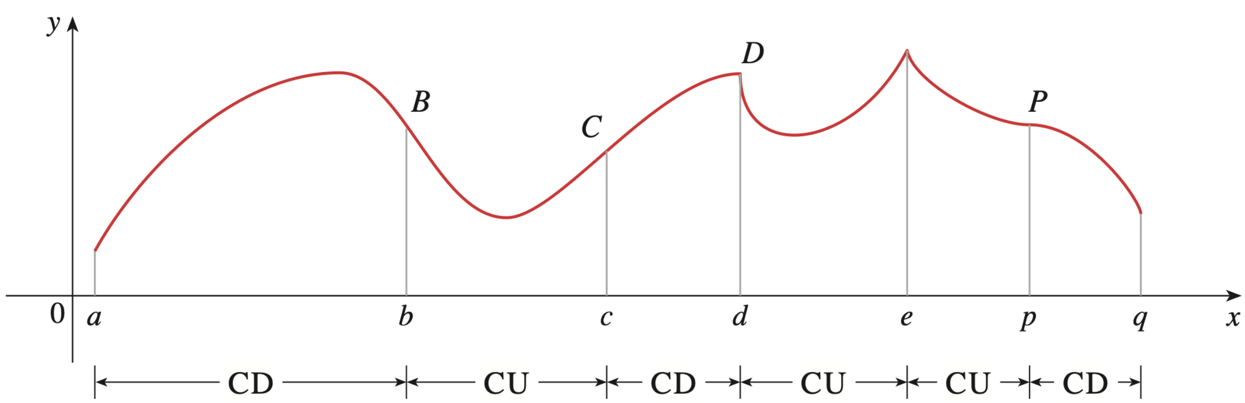
Theorem (Concavity Test)**.** Let f be a function such that f'' exists for all x in I.

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

(c) If $f''(c) = 0$ at a point c, no conclusion about concavity can be drawn from this alone (the test is inconclusive at c).

Definition (Inflection Point)**.** A point P on the curve $y = f(x)$ is called an *inflection point* if f is continuous at P and the curve changes concavity at P.



Theorem (Second Derivative Test)**.** Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither.

Note: If $f''(c)$ does not exist, the test cannot be applied.

Example. Analyze the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example. Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.