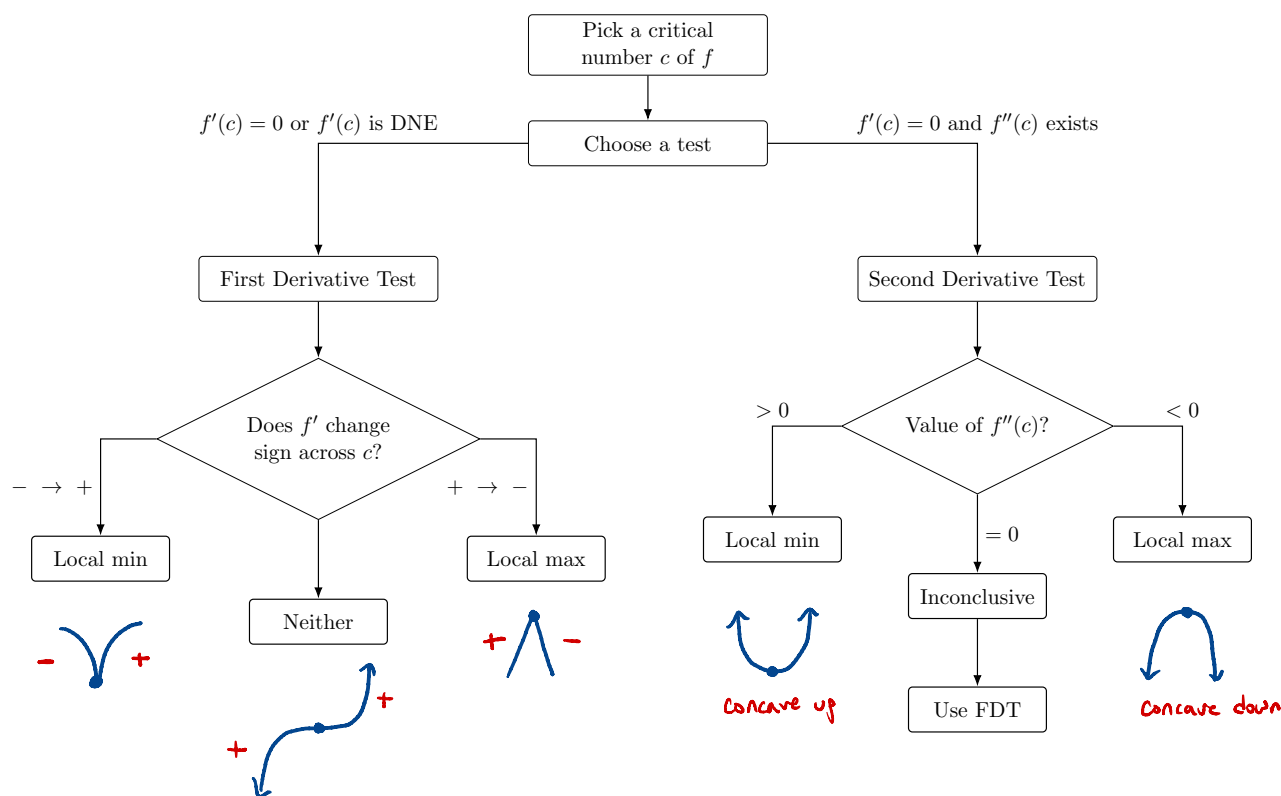


4.3 / 4.5 Curve Sketching

The main goal of this section is to decide what happens at a function's *critical numbers*—points in the domain where $f'(x) = 0$ or $f'(x)$ does not exist. For each such point, we want to determine whether f has a local maximum, a local minimum, or neither.



Combining this information with what we already know about increasing/decreasing and concavity will allow us to accurately analyze and sketch curves.

Theorem (Increasing/Decreasing Test).

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval. *
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof.

1. **Fix points.** Pick any x_1, x_2 in I with $x_1 < x_2$. According to the definition of “increasing,” what do we need to prove?

$$f(x_1) < f(x_2)$$

2. **Prepare to use MVT.** Because $f'(x) > 0$ on I , the function f is differentiable on (x_1, x_2) and hence continuous on $[x_1, x_2]$. By the MVT, there is some c in (x_1, x_2) so that

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

$$\text{MVT: } f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

3. **Sign analysis of the right-hand side.** Using the hypotheses $f'(x) > 0$ on I and $x_2 > x_1$, fill in the inequalities:

$$f'(c) > 0 \quad \text{and} \quad x_2 - x_1 > 0.$$

Therefore the product $f'(c)(x_2 - x_1)$ is also > 0 .

4. **Translate back to $f(x_2) - f(x_1)$.** We conclude that

$$f(x_2) - f(x_1) > 0.$$

[since $f(x_2) - f(x_1) = \text{pos \#}$]

as well, so

$$f(x_1) < f(x_2)$$

We conclude that f is increasing on I .

□

Example. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

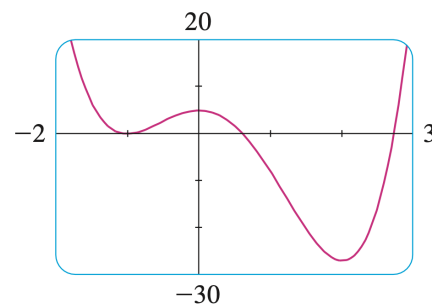
- find where f' is positive and where f' is negative

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x-2)(x+1)$$

← Critical #s are -1, 0, and 2



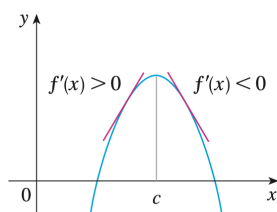
- Create a sign chart, splitting the domain at the critical #s (f can only change signs at these values)

	$12x$	$x-2$	$x+1$	$f'(x)$	$f(x)$
$x < -1$	-	-	-	-	Decreasing
$-1 < x < 0$	-	-	+	+	Increasing
$0 < x < 2$	+	-	+	-	Decreasing
$x > 2$	+	+	+	+	Increasing

Theorem (First Derivative Test). Let c be a critical number of a continuous function f .

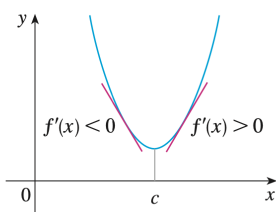
- (a) If f' changes from positive to negative at c , then f has a local maximum at c .
- (b) If f' changes from negative to positive at c , then f has a local minimum at c .
- (c) If f' is positive to the left and right of c , or negative to the left and right of c , then f has no local maximum or minimum at c .

Example. Visualizing the first derivative test.



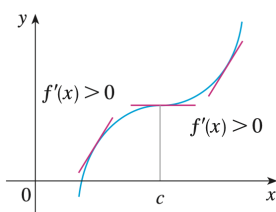
(a) Local maximum

f' changes from
+ to -



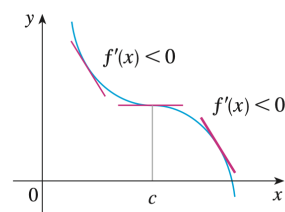
(b) Local minimum

f' changes from
- to +



(c) No maximum or minimum

f' doesn't change sign



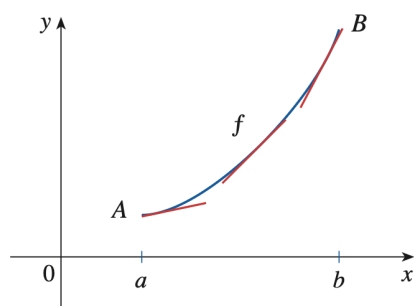
(d) No maximum or minimum

Example. Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

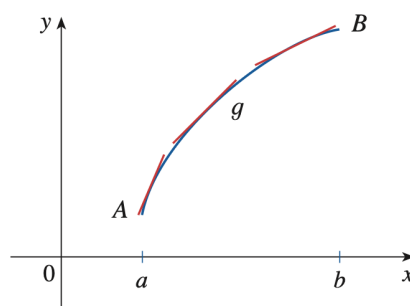
Example. Find the local maximum and minimum values of the function $g(x) = x + 2 \sin x$ on the interval $[0, 2\pi]$

Definition (Concavity). Let I be an interval and $f(x)$ be differentiable on I .

- (a) f is concave upward on I if, for every x in I , the graph of f lies above its tangent line at x .
- (b) f is concave downward on I if, for every x in I , the graph of f lies below its tangent line at x .



(a) Concave upward

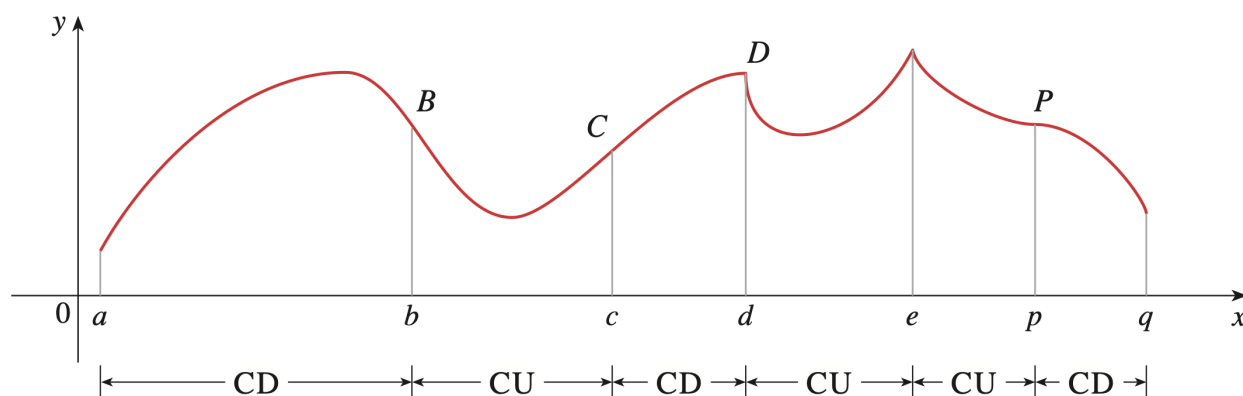


(b) Concave downward

Theorem (Concavity Test). Let f be a function such that f'' exists for all x in I .

- (a) If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .
- (b) If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .
- (c) If $f''(c) = 0$ at a point c , no conclusion about concavity can be drawn from this alone (the test is inconclusive at c).

Definition (Inflection Point). A point P on the curve $y = f(x)$ is called an *inflection point* if f is continuous at P and the curve changes concavity at P .



Theorem (Second Derivative Test). Suppose f'' is continuous near c .

- (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
- (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .
- (c) If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither.

Note: If $f''(c)$ does not exist, the test cannot be applied.

Example. Analyze the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example. Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

