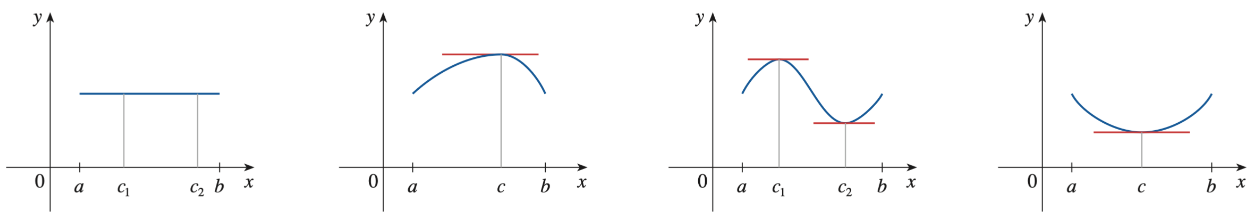
4.2 The Mean Value Theorem

Theorem (Rolle's Theorem)**.** Let f be a function that satisfies the following:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

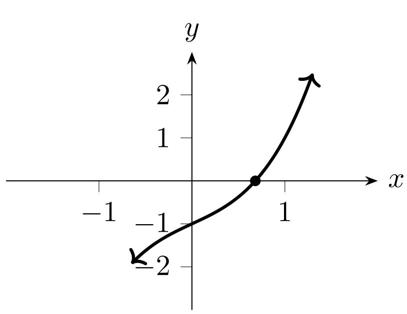
Then there is a number c in (a, b) such that $f'(c) = 0$.



(a) (b) (c) (d)

Example. If an object is in the same place at two different instants $t = a$ and $t = b$, what does Rolle's Theorem say about the velocity of the object?

Example. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real solution.



Theorem (Mean Value Theorem)**.** Let f be a function that satisfies the following:

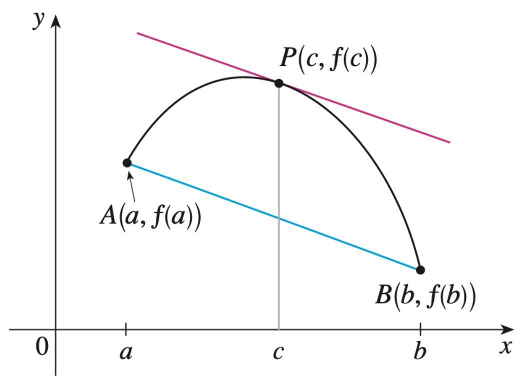
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

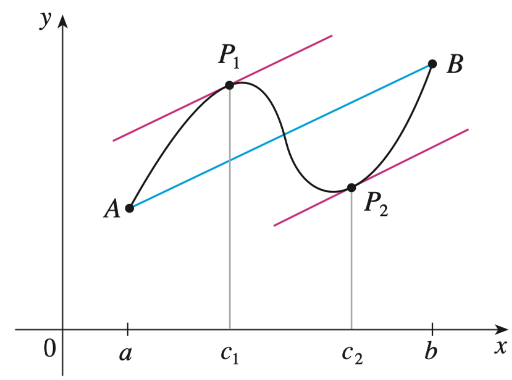
Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

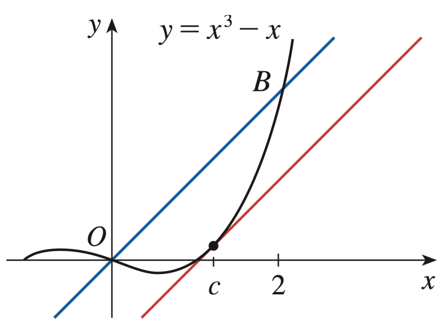
Equivalently,

$$f(b) - f(a) = f'(c)\,(b - a).$$





Example. Does the function $f(x) = x^3 - x$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?



Example. What does the Mean Value Theorem say about velocity?

Example. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ be?

Theorem. If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof.

1. Pick any two points x_1 ____ x_2 in ______________. Goal: show $f(x_1) =$ ___________.

2. Since $f(x)$ is differentiable on (hence continuous) on (a, b), the MVT applies. So there exists c in ______________ such that

$$f'(c) = \underline{\hspace{6cm}} \qquad \text{or} \qquad f(x_2) - f(x_1) = \underline{\hspace{7cm}}.$$

3. Hence:

$$f(x_2) - f(x_1) = \underline{\hspace{3cm}} \cdot (x_2 - x_1) = \underline{\hspace{3cm}},$$

hence $f(x_2) =$ ______________________.

4. Since x_1, x_2 were arbitrary, f has the same value at any two points of (a, b). Therefore f is __________________ on _________________.

□

Corollary. If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Proof.

□

Example. For the function

$$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

$f'(x) = 0$ for all x in the domain. Why doesn't this contradict the theorem?

Example. Prove the identity $\tan^{-1}(x) + \cot^{-1}(x) = \dfrac{\pi}{2}$.