• In 4.1 we saw how to find ABSOLUTE max/mins

• Current goal: classify critical #'s as local max, local min, or neither.

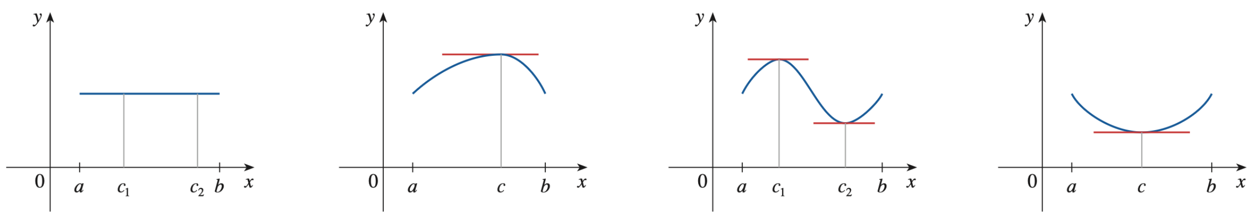
4.2 The Mean Value Theorem

• Detour: Need MVT to prove desired theorems

Theorem (Rolle's Theorem)**.** Let f be a function that satisfies the following:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.



(a) (b) (c) (d)

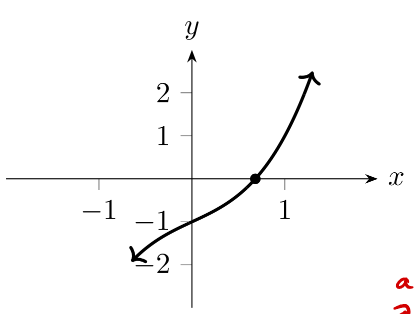
In each example, f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, and so $f'(c) = 0$ for some point c in (a,b).

Example. If an object is in the same place at two different instants $t = a$ and $t = b$, what does Rolle's Theorem say about the velocity of the object?

If $f(t)$ is position with $f(a) = f(b)$ then at some point in (a,b) the velocity is 0.

(Think about a ball thrown upward ... what goes up must come down)

Example. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real solution.



Let $f(x) = x^3 + x - 1$. Then $f(0) < 0$ and $f(1) > 0$.

Since $f(x)$ is continuous, the I.V.T. applies.

So $f(x) = 0$ at some c in $(0,1)$.

a and b

What if there were two solutions? Then $f(a) = f(b) = 0$. By Rolle's Thm, there is some c in (a,b) with $f'(c) = 0$. But $f'(x) = 3x^2 + 1 \geq 1$ for all x.

Contradiction $\rightarrow\leftarrow$. Conclude: there is just one solution.

Rolle's Thm is needed in the proof of MVT

Theorem (Mean Value Theorem)**.** Let f be a function that satisfies the following:

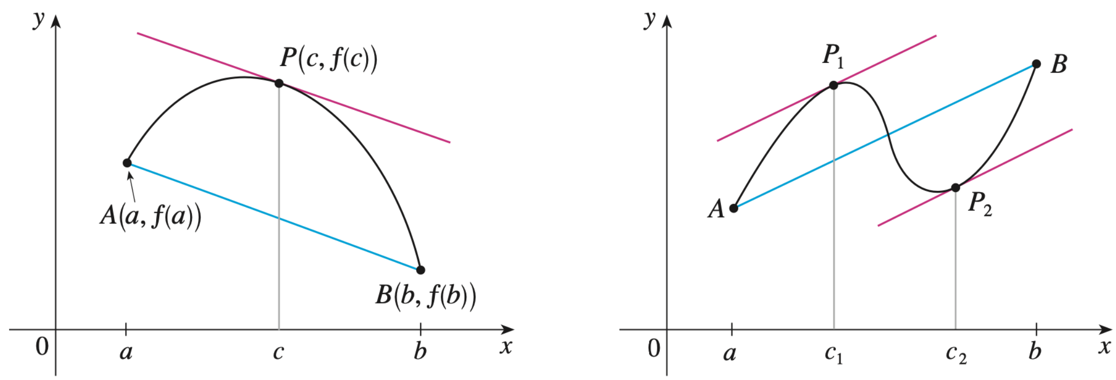
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

the derivative at c → $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ ← the average rate of change of f(x) on (a,b)

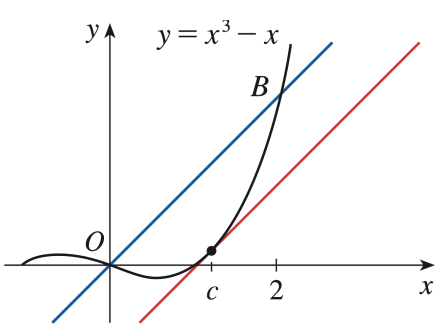
Equivalently,

$$f(b) - f(a) = f'(c)\,(b - a).$$



Slope of blue line is the avg. rate of change. Red line is at the point whose tangent is parallel to the blue line. MVT says such a red line always exists.

Example. Does the function $f(x) = x^3 - x$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?



Yes. $f(x) = x^3 - x$ is a polynomial and so is continuous on $[0,2]$ ✓ and differentiable on $(0,2)$ ✓

and so for some c in $(0,2)$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2 - 0} = 3$$

$f'(x) = 3x^2 - 1$

$\Rightarrow 3c^2 - 1 = 3$

$\Rightarrow 3c^2 = 4$

$\Rightarrow c^2 = 4/3$

$\Rightarrow c = \pm\sqrt{4/3}$

$\Rightarrow c = \pm 2/\sqrt{3}$

$\Rightarrow \boxed{c = 2/\sqrt{3}}$

Example. What does the Mean Value Theorem say about velocity?

Over any interval, there is some point at which the instantaneous velocity equals the average velocity on the interval.

e.g. If a car traveled 180 km in 2 hours, then the speedometer must have said 90 km/hr at least once.

Example. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ be?

Apply the MVT to $[0,2]$. For some c,

$$f'(c) = \frac{f(2) - f(0)}{2-0}$$

$$\Rightarrow f(2) - f(0) = 2 \cdot f'(c)$$

$$\Rightarrow f(2) = 2f'(c) + f(0)$$

For all c, $f'(c) \leq 5$

$$\Rightarrow f(2) \leq 2 \cdot 5 + f(0)$$

$$\Rightarrow f(2) \leq 2 \cdot 5 + (-3)$$

$$\Rightarrow f(2) \leq 7$$

Theorem. If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof.

1. Pick any two points x_1 $\underline{\ <\ }$ x_2 in $\underline{(a,b)}$. Goal: show $f(x_1) = \underline{f(x_2)}$.

2. Since $f(x)$ is differentiable on (hence continuous) on (a, b), the MVT applies. So there exists c in $\underline{(x_1, x_2)}$ such that

$$f'(c) = \underline{\frac{f(x_2)-f(x_1)}{x_2-x_1}} \qquad \text{or} \qquad f(x_2) - f(x_1) = \underline{f'(c)\cdot(x_2-x_1)}.$$

3. Hence:

$$f(x_2) - f(x_1) = \underline{0} \cdot (x_2 - x_1) = \underline{0},$$

hence $f(x_2) = \underline{f(x_1)}$.

4. Since x_1, x_2 were arbitrary, f has the same value at any two points of (a, b). Therefore f is $\underline{\text{Constant}}$ on $\underline{(a,b)}$.

□

Corollary. If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Proof.

Let $F(x) = f(x) - g(x)$. Then $F'(x) = f'(x) - g'(x) = 0$ on (a,b)

Hence $F(x)$ is constant by the Thm. Hence $f(x) - g(x)$ is constant.

□

Example. For the function

$$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

$f'(x) = 0$ for all x in the domain. Why doesn't this contradict the theorem?

The domain is not an interval (a,b)

Example. Prove the identity $\tan^{-1}(x) + \cot^{-1}(x) = \dfrac{\pi}{2}$.

Let $f(x) = \tan^{-1}(x) + \cot^{-1}(x)$

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Hence $f(x) = C$, a constant

Since $f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

the constant is $C = \frac{\pi}{2}$

Conclude: $f(x) = \tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$ ✓