4.1 Maximum and Minimum Values

Definition. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- Absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- Absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition. Let c be a number in the domain D of a function f. Then $f(c)$ is a

- Local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- Local minimum value of f if $f(c) \leq f(x)$ when x is near c.

"Near c" means on some open interval I containing c. In particular, a local maximum or minimum cannot occur at an endpoint of an interval.

Example. The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leq x \leq 4$$

is graphed below. Find any maxima or minima.

Absolute maximum at $(-1, 37)$

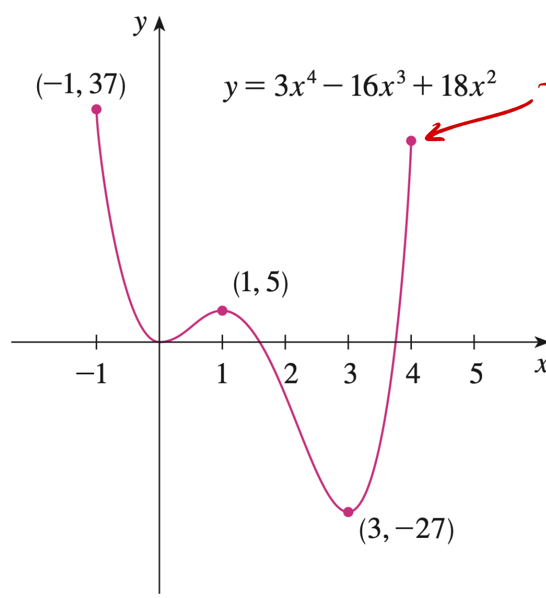
Not a local max since it is at an endpoint

Absolute minimum at $(3, -27)$

This is a local minimum

Local minimum at $(0,0)$

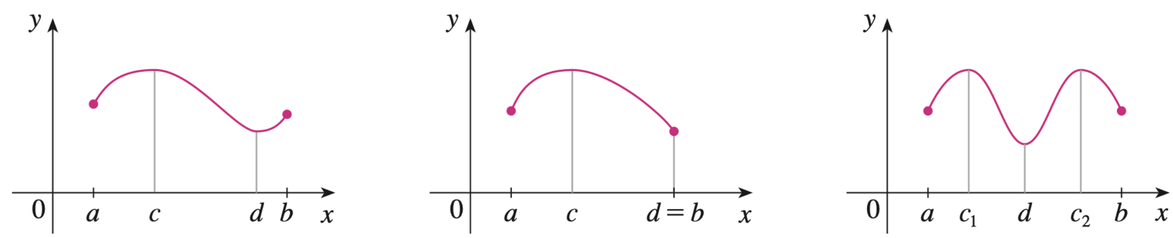
Local maximum at $(1,5)$



This is nothing

Theorem (Extreme Value Theorem)**.** If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Example.

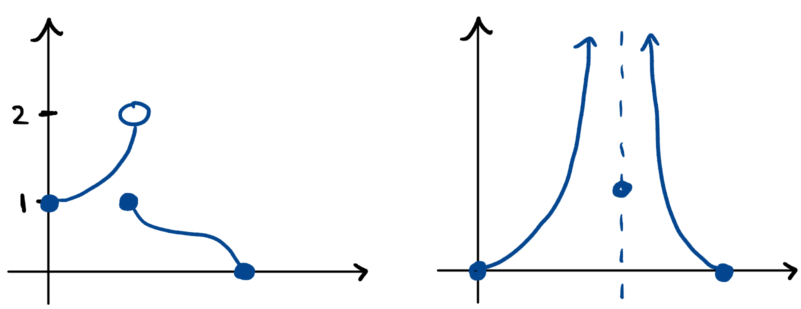


The absolute max and absolute min occur on the inside of [a, b]. Both of these are local extrema.

Absolute/local max at c and absolute min at endpoint (not a local min)

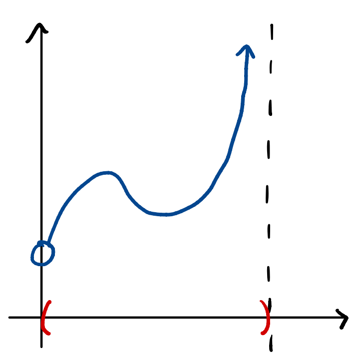
We can have more than one point be the absolute max (or min).

Example. Are absolute extrema guaranteed if the function *not continuous* on the closed interval?



These functions have an absolute minimum but no absolute maximum.

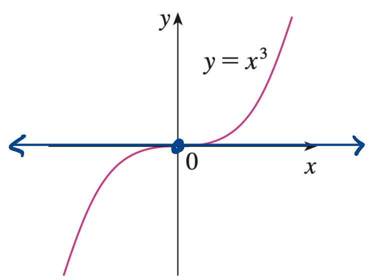
Example. Are absolute extrema guaranteed for a continuous function on an *open interval*?

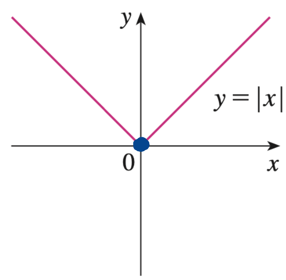


This function has neither an absolute minimum nor an absolute maximum.

Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example. Why do we have to be careful with the above theorem?





$f'(c) = 0$ doesn't mean f has a local max/min at c necessarily

$f(x)$ has a local minimum at 0, but $f'(0)$ is D.N.E (corner).

Definition. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

* At a local max or min the derivative is either DNE or 0

Therefore, local max/mins <u>can only occur</u> at critical numbers.

(Critical numbers don't have to be local max/mins)

Example. Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

$$f'(x) = x^{3/5}(-1) + (4-x)\cdot \frac{3}{5}x^{-2/5}$$

$$= \frac{12-8x}{5x^{2/5}}$$

- This is 0 when $12 - 8x = 0$ (and $5x^{2/5} \neq 0$) $\Rightarrow x = 3/2$
- This is DNE when $x = 0$ (can't divide by 0)

$$\boxed{\text{Critical \#'s: } 0 \text{ and } \frac{3}{2}}$$

↑

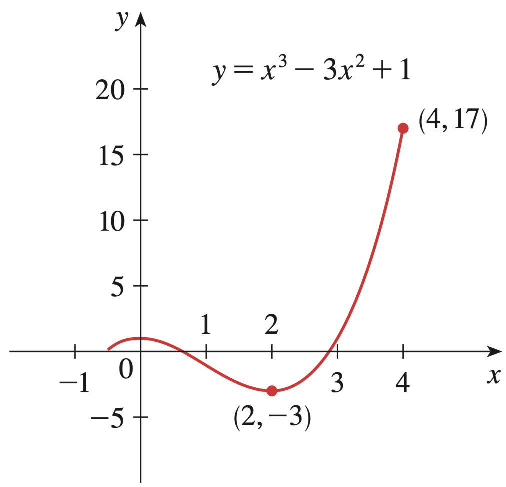
Any local max/min would have to be at one of these.

This is possible because of the Extreme Value Theorem

The Closed Interval Method. To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the critical numbers.
2. Find the values of f at the critical numbers of f in (a, b). All possible local extrema
3. Find the values of f at the endpoints of the interval. Local extrema can't occur here but absolute extrema can
4. The largest of the values from Steps 1-3 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example. Find the absolute minimum and maximum values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.



① $f'(x) = 3x^2 - 6x$

This exists everywhere on $[-\frac{1}{2}, 4]$

This is 0 when $3x(x-2) = 0$

i.e. when $x=0$ or $x=2$

$\boxed{\text{critical \#'s: } 0 \text{ and } 2}$

② $f(0) = 1$

$f(2) = -3$

All possible locations for local max/mins

③ $f(-\frac{1}{2}) = \frac{1}{8}$

$f(4) = 17$

④ Absolute max at $(4, 17)$ Endpoint!

Absolute min at $(2, -3)$ Also a local min

Example. Find the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$ on the interval $[0, 2\pi]$.

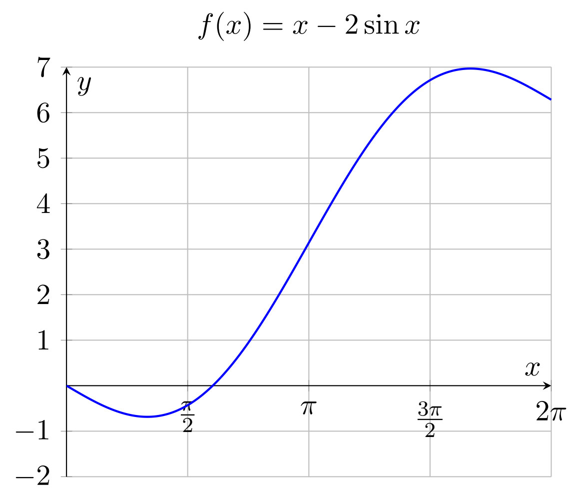
(1) $f'(x) = 1 - 2\cos(x)$

This exists everywhere on $[0, 2\pi]$

This is 0 when:

$$2\cos(x) = 1$$
$$\cos(x) = \frac{1}{2}$$
$$x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$\boxed{\text{Critical \#'s: } \frac{\pi}{3} \text{ and } \frac{5\pi}{3}}$



(2) $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = -0.685$

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = 6.97$

All possible locations for local max/mins

(3) $f(0) = 0$ and $f(2\pi) = 2\pi \approx 6.28$

(4) Absolute max at $\left(\frac{5\pi}{3}, 6.97\right)$

Absolute min at $\left(\frac{\pi}{3}, -0.685\right)$