3.9 Related Rates

In related rates problems, two or more quantities are changing with respect to time. The goal is to use a geometric or physical relation (area, volume, Pythagoras, trig, etc.), differentiate with respect to t, and connect the given rate(s) to the unknown rate.

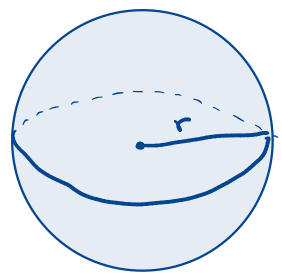
Step-by-step process:

1. Draw a clear diagram and label all changing quantities with variables. Write down the given information and what you want.
2. Write an equation relating those variables (geometry, trig, physics).
3. Differentiate implicitly with respect to t.
4. Substitute all known values at the instant of interest and solve for the desired rate.

Example. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the radius is 25 cm?

① Given: $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Want: $\frac{dr}{dt}$ when $r = 25$ cm



② $V = \frac{4}{3}\pi r^3$ [Volume of a sphere]

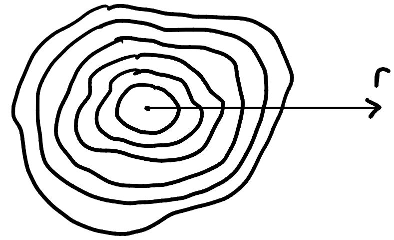
chain rule

③ $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

④ $100 = 4\pi (25)^2 \cdot \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\cancel{100}}{\cancel{4}\pi \cdot \cancel{25} \cdot 25} = \frac{1}{25\pi} \approx \boxed{0.0127 \text{ cm/s}}$$

Example. A stone dropped in a pond sends out a circular ripple whose radius increases at a constant rate of 4 ft/s. After 12 seconds, how rapidly is the area enclosed by the ripple increasing?



① Given: $\frac{dr}{dt} = 4$ ft/s

Want: $\frac{dA}{dt}$ when $t = 12$ sec

② $A = \pi r^2$ [area of a circle]

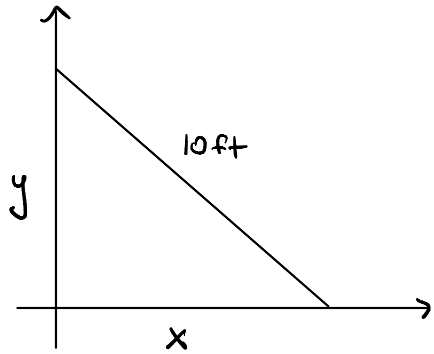
③ $\frac{dA}{dt} \overset{\text{chain rule}}{=} \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot \frac{dr}{dt}$

④ $\frac{dA}{dt} = 2\pi r \cdot 4$ At $t = 12$ s, $r = 4 \cdot 12 = 48$ ft

$\frac{dA}{dt} = 2\pi \cdot 48 \cdot 4$

$$\boxed{\frac{dA}{dt} = 384\pi \text{ ft}^2/\text{sec}}$$

Example. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom is 6 ft from the wall?



① Given: $\frac{dx}{dt} = 1$

Want: $\frac{dy}{dt}$ when $x = 6$

② $x^2 + y^2 = 100$ [Pythagorean Thm]

③ $2x \cdot \underset{\text{chain rule}}{\frac{dx}{dt}} + 2y \cdot \underset{\text{chain rule}}{\frac{dy}{dt}} = 0$

④ $2 \cdot 6 \cdot 1 + 2y \cdot \frac{dy}{dt} = 0$

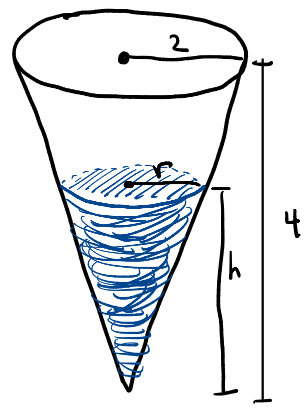
What is y? use $x^2+y^2=100$ when $x=6$, $y=8$

$12 + 2 \cdot 8 \frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-12}{16} = \boxed{\frac{-3}{4} \text{ ft/sec}}$

The speed at which the ladder is falling is $\boxed{3/4 \text{ ft/sec}}$

Example. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

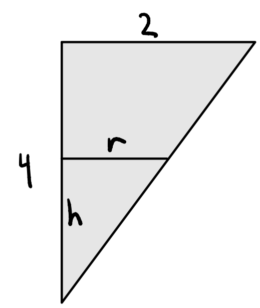


1 Given: $\dfrac{dV}{dt} = 2 \text{ m}^3/\text{min}$

Want: $\dfrac{dh}{dt}$ when $h = 3$ m

2 $V = \frac{1}{3}\pi r^2 \cdot h$ [volume of a cone]

Need V as a function of h alone. Eliminate r.



By similar triangles, $\dfrac{r}{h} = \dfrac{2}{4} \Rightarrow r = \frac{1}{2}h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3$

3 $\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt} = \dfrac{\pi}{4}h^2 \cdot \dfrac{dh}{dt}$

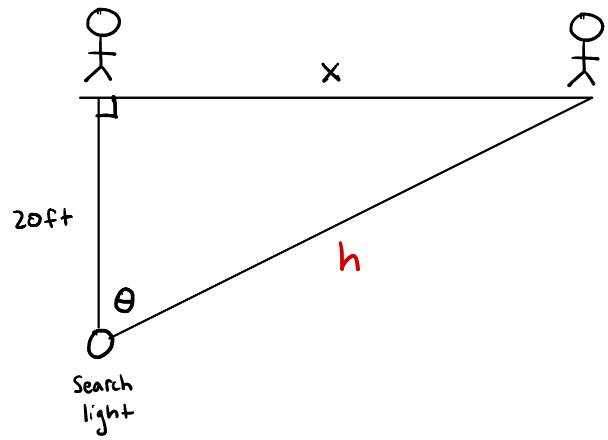
4 $2 = \dfrac{\pi}{4} \cdot (3)^2 \cdot \dfrac{dh}{dt}$

$\dfrac{dh}{dt} = \dfrac{2}{9} \cdot \dfrac{4}{\pi} = \dfrac{8}{9\pi} \approx \boxed{0.28 \text{ m/min}}$

Example. A boat is sailing north at 15 mph. At the same instant, a patrol boat is 9 miles east of it and sailing east at 20 mph. At what rate is the distance between the two boats changing after 1 hour?

Example. A street lamp is 12 ft tall. A person who is 6 ft tall walks away from the base of the lamp at a speed of 3 ft/s. How fast is the tip of the shadow moving when the person is 20 ft from the lamp?

Example. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



① Given: $\frac{dx}{dt} = 4 \text{ ft/s}$

Want: $\frac{d\theta}{dt}$ when $x = 15\text{ft}$

② $\tan(\theta) = \frac{x}{20}$

$x = 20\tan(\theta)$

③ $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = 20\sec^2(\theta) \cdot \frac{d\theta}{dt}$

④ $4 = 20\sec^2(\theta) \cdot \frac{d\theta}{dt}$

$4 = 20 \cdot \frac{1}{(4/5)^2} \cdot \frac{d\theta}{dt}$

$\frac{4}{20} \cdot \left(\frac{4}{5}\right)^2 = \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{16}{25} = \boxed{\frac{16}{125} \text{ rad/sec}}$

What is $\sec^2(\theta)$ when $x = 15\text{ft}$?

$\sec^2(\theta) = \frac{1}{\cos^2(\theta)}$

$\cos(\theta) = \frac{20}{h}$

When $x = 15$, $h^2 = 15^2 + 20^2$

implies $h = 25$

$\cos(\theta) = \frac{20}{25} = \frac{4}{5}$

Example. A water trough is 10 m long. Its cross-section is an isosceles trapezoid that is 2 m wide at the bottom, 6 m wide at the top, and 4 m tall. Water is being poured in at a constant rate of 1 m^3/min. How fast is the water level rising when the water is 2 m deep?

Example. A water wheel has radius 10 m and makes one full revolution every 2 minutes. How fast (in m/min) is a drop of paint on the rim *rising* when it is 18 m above the bottom of the wheel?