3.7 Rates of Change in the Natural and Social Sciences

The derivative appears in many different contexts:

Physics
- Velocity
- Acceleration
- Electric current
- Temperature change

Chemistry
- Reaction rate
- Gas volume change
- Temperature change in a reaction
- pH change per volume

Biology/Medicine
- Population growth
- Drug concentration change
- Heart-rate change
- Bacterial growth

Economics/Business
- Marginal cost
- Marginal revenue
- Marginal profit
- Sales growth

Sociology/Epidemiology
- New cases per day
- Adoption rate
- Population change
- Info spread

Psychology/Education
- Learning rate
- Forgetting/decay
- Reaction-time improvement
- Performance change

Earth Science/Geology
- Sea-level rise
- River stage change
- Erosion rate
- Lapse rate

Engineering/Technology
- Speed of a mechanism
- Fuel/battery usage
- Pipe flow rate
- Charge/discharge

Environmental/Climate
- CO_2 trend
- Air-quality change
- Lake volume change
- Ice/snow mass loss

"One idea... many interpretations."

The applications we focus on are:
- Velocity and acceleration
- Population growth
- Marginal cost

Example. The position of a particle is given by

$$s = f(t) = t^3 - 6t^2 + 9t,$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

(a) $v(t) = \dfrac{ds}{dt} = 3t^2 - 12t + 9$

(b) $v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$

$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$

(c) The particle is at rest when $v(t) = 0$

$$3t^2 - 12t + 9 = 0$$

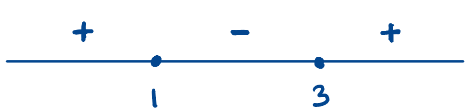
$$3(t^2 - 4t + 3) = 0$$

$$3(t-1)(t-3) = 0$$

$t = 1$ sec or $t = 3$ sec

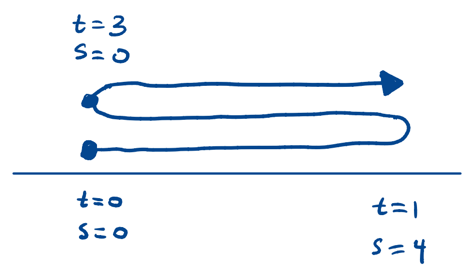
(d) Moving forward means $v(t) > 0$

$$3(t-1)(t-3) > 0$$



$t < 1$ sec or $t > 3$ sec

(e)



(f) Calculate $[0,1]$, $[1,3]$, $[3,5]$ separately

$$\left.\begin{aligned} |f(1) - f(0)| &= |4 - 0| = 4\text{m} \\ |f(3) - f(1)| &= |0 - 4| = 4\text{m} \\ |f(5) - f(3)| &= |20 - 0| = 20\text{m} \end{aligned}\right\} \text{Total} = 28\text{m}$$

(g) $a(t) = \dfrac{d^2s}{dt^2} = \dfrac{dv}{dt} = 6t - 12$

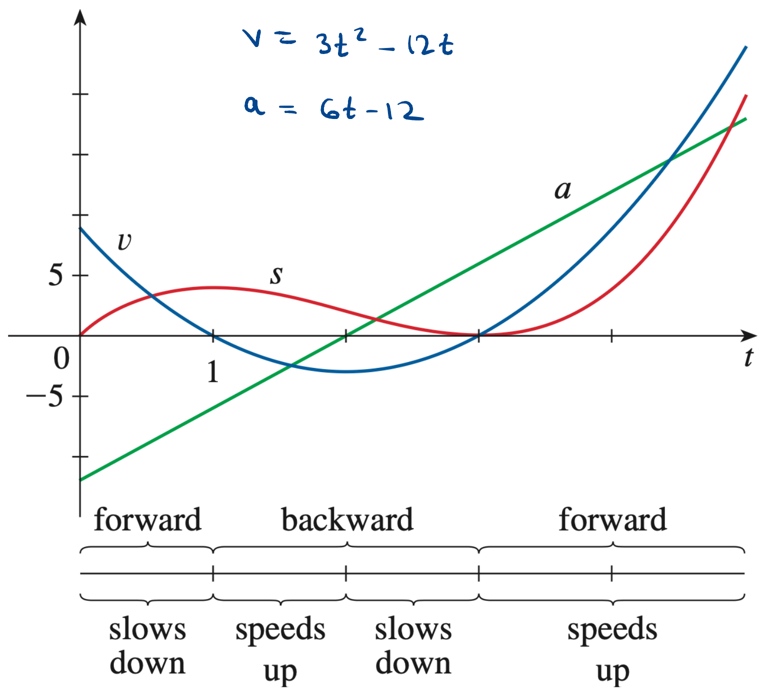
$a(4) = 6 \cdot 4 - 12 = 12 \text{ m/s}^2$

(h) and (i)

$S = t^3 - 6t^2 + 9t$

$v = 3t^2 - 12t$

$a = 6t - 12$



Speed up: velocity positive and increasing
velocity negative and decreasing
(velocity and acceleration have the same sign)

Slow down: velocity positive and decreasing
velocity negative and increasing
(velocity and acceleration have opposite signs)

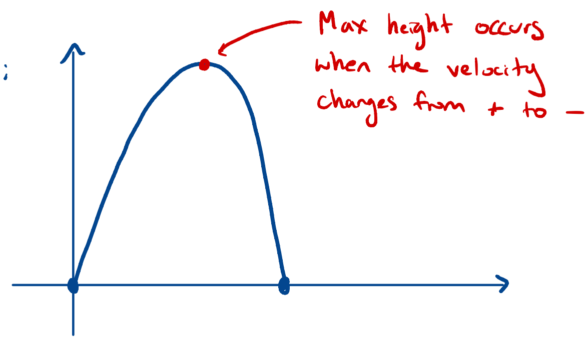
Example. If a ball is thrown straight up with initial velocity 128 ft/s, its height after t seconds is

$$s(t) = 128t - 16t^2.$$

(a) What is the maximum height reached by the ball?

(b) What is the velocity of the ball when it is 192 ft above the ground on its way up? On its way down?

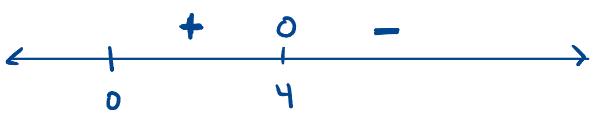
(a) Idea:



$V(t) = 128 - 32t$

$v(t) = 0$ when $128 = 32t \Rightarrow t = 4$

$V(t)$



sign chart

Since position increases for $0 < t < 4$ (velocity > 0) and decreases for $t > 4$ (velocity < 0), the maximum height occurs at $t = 4$, and it is $s(4) = 256$ m

(b) Solve $s(t) = 192$

$128t - 16t^2 = 192$

$-16(t^2 - 8t + 12) = 0$

$t^2 - 8t + 12 = 0$

$(t-2)(t-6) = 0$

$t = 2$ or $t = 6$

On way up:

$v(2) = 128 - 32(2) = 64$ ft/s

On way down:

$v(6) = 128 - 32(6) = -64$ ft/s

Exponential Models. All three are the same model written in different ways. Pick the one that matches the wording and units.

- **Continuous rate given:**
$$P(t) = Ce^{kt}$$
Use this when the wording says *"grows continuously at rate k"*.

- **Per-unit multiplier given:**
$$\boxed{P(t) = Ca^t} \quad \star$$
Use when you're told "multiplies by a each (same) time unit," e.g., "doubles every year."

- **Per-n-units multiplier given:**
$$P(t) = Ca^{t/n}$$
Use when the multiplier applies every n time units, e.g., "triples every 5 years" corresponds to $a = 3$, $n = 5$, t in years.

Example. A population of bacteria quadruples every hour and begins with 150 bacteria.

(a) Find a formula for the number $P(t)$ of bacteria after t hours.

(b) Use it to estimate the rate of growth at $t = 1.5$ hours.

(a) Quadruples every hour $\Rightarrow$ $P(t) = 150 \cdot 4^t$

(b) $P'(t) = \frac{d}{dt}(150 \cdot 4^t) = 150 \frac{d}{dt}(4^t) = 150 \cdot 4^t \cdot \ln(4)$

$$P'(1.5) = 150 \cdot 4^{1.5} \cdot \ln(4)$$
$$= 150 \cdot 4^{3/2} \cdot \ln(4)$$
$$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$$
$$= 150 \cdot 8 \cdot \ln(4)$$
$$= 1200 \ln(4) \approx 1663.6 \text{ bacteria/hour}$$

Cost and Marginal Cost. Let $C(x)$ be the total cost (dollars) to produce x units. The *marginal cost* is the derivative

$$C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h},$$

which gives the instantaneous rate of change of cost with respect to output (units: \$/unit).

$$\underbrace{C'(x)}_{\text{instantaneous slope at } x} \qquad \text{vs.} \qquad \underbrace{C(x+1) - C(x)}_{\text{exact extra dollars for the next unit}} .$$

- $C'(x)$ describes how cost is changing *right at* output x; it is a slope at a point.
- $C(x+1) - C(x)$ is the actual increase in cost to go from x to $x+1$ units.
- When the cost curve is fairly straight on $[x, x+1]$, these are close and we use

$$C(x+1) - C(x) \approx C'(x).$$

In other words, $C'(x)$ is used to *estimate* the cost of the next unit.

Example. A company makes reusable stainless-steel water bottles. The total cost (in dollars) to produce x bottles is

$$C(x) = 1500 + 4x + 0.03x^2 + 0.0004x^3.$$

(a) Find the marginal cost function $C'(x)$.

(b) Compute $C'(120)$. What does this number predict?

(c) Find the *actual* cost of manufacturing the 121^{st} bottle.