MATH 1300 — Derivatives of Logarithmic Functions (Solutions)

1. Compute each exactly.
   (a) $\log_2\left(64^{3/2}\right)$

   Solution: $\log_2(64^{3/2}) = \frac{3}{2}\log_2(64) = \frac{3}{2}\cdot 6 = 9$.

   (b) $\log_4\left(\dfrac{1}{256}\right)$

   Solution: $\frac{1}{256} = 4^{-4}$, so $\log_4\left(\frac{1}{256}\right) = -4$.

   (c) $\ln\left(e^{-5}\right)$

   Solution: $\ln(e^{-5}) = -5$.

2. Differentiate $p(t) = \ln\left(t^2 + 8\right)$.

   Solution:
   $$p'(t) = \frac{2t}{t^2+8}.$$

3. Differentiate $y = \log_5\left((x^2 - 3x)^4\right)$.

   Solution:
   $$y = 4\log_5(x^2 - 3x) \quad \Rightarrow \quad y' = \frac{4(2x-3)}{(x^2-3x)\ \ln 5}.$$
   (Domain: $x^2 - 3x > 0$, $x \neq 0, 3$.)

4. Let $f(x) = \log_b(7x+2)$ with $b > 0$, $b \neq 1$. Find b such that $f'(0) = 5$.

   Solution:
   $$f'(x) = \frac{7}{(7x+2)\ \ln b} \ \Rightarrow\ f'(0) = \frac{7}{2\ln b} = 5.$$
   Thus $\ln b = \dfrac{7}{10}$ and
   $$b = e^{7/10}.$$

5. Compute y' for $y = \big(\sin(2x)\big)^x$.

Solution: Take ln:

$$\ln y = x\ln(\sin 2x).$$

Differentiate:

$$\frac{y'}{y} = \ln(\sin 2x) + x \cdot \frac{2\cos 2x}{\sin 2x} = \ln(\sin 2x) + 2x\cot(2x).$$

Therefore

$$y' = \big(\sin 2x\big)^x \Big[\ln(\sin 2x) + 2x\cot(2x)\Big].$$

6. Compute y' for

$$y = \big(x^2+1\big)^{\sin x}.$$

Solution:

$$\ln y = (\sin x)\ \ln(x^2+1).$$

Differentiate:

$$\frac{y'}{y} = (\cos x)\ \ln(x^2+1) + (\sin x)\,\frac{2x}{x^2+1}.$$

Hence

$$y' = \big(x^2+1\big)^{\sin x}\left[\cos x\ \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right].$$

7. Compute y' for

$$y = x^{x^2} \quad (x>0).$$

Solution:

$$\ln y = x^2\ln x \quad \Rightarrow \quad \frac{y'}{y} = 2x\ln x + x.$$

Therefore

$$y = x^{x^2}\big(2x\ln x + x\big) = x^{x^2+1}\big(2\ln x + 1\big).$$