3.6 Inverse Trigonometric Functions and Their Derivatives

Need to know

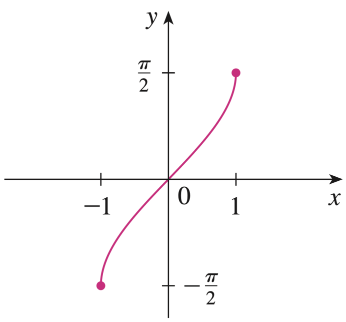
$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2},$$

Won't be tested on these

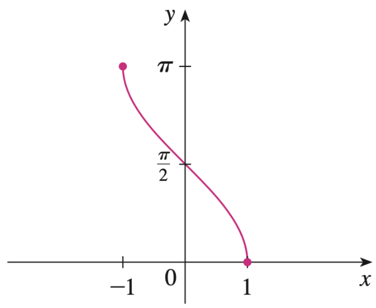
$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}.$$

arcsin(x) arccos(x) arctan(x) ⚠ $\sin^{-1}(x)$ is the inverse function. It is <u>not</u> $\frac{1}{\sin(x)}$.

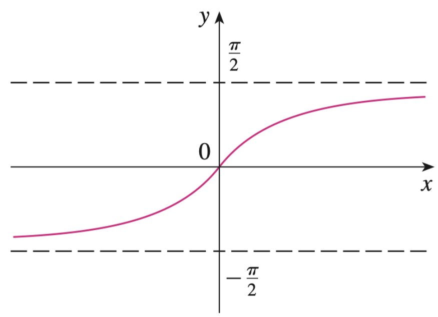
Question. What are the functions $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$? What are the domains and ranges of these functions?



$f(x) = \arcsin(x)$



$f(x) = \arccos(x)$



$f(x) = \arctan(x)$

input: a value x in $[-1, 1]$

output: an angle θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = x$

input: a value x in $[-1, 1]$

output: an angle θ in $[0, \pi]$ with $\cos(\theta) = x$

input: a value x in $(-\infty, \infty)$

output: an angle θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $\tan(\theta) = x$

Example. Evaluate

(a) $\sin^{-1}(\frac{1}{2})$

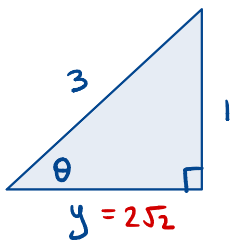
(b) $\tan(\arcsin(\frac{1}{3}))$

(a) We want an angle θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = \frac{1}{2}$

$\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\frac{\pi}{6}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\boxed{\sin^{-1}(\tfrac{1}{2}) = \frac{\pi}{6}}$$

(b)



$\sin(\theta) = \frac{1}{3}$

$\theta = \arcsin(\frac{1}{3})$ [because θ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$]

By Pythagorean Thm, $y = \sqrt{3^2 - 1^2} = 2\sqrt{2}$

$\tan(\arcsin(\frac{1}{3})) = \tan(\theta) = \frac{1}{2\sqrt{2}}$ $\frac{\text{opp}}{\text{adj}}$

Theorem. Show that $\dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof.

let $y = \sin^{-1}(x)$

$\Rightarrow \sin(y) = x$

$\Rightarrow \frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$

$\Rightarrow \cos(y) \cdot \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} \underset{\cos^2(y)+\sin^2(y)=1}{=} \frac{1}{\sqrt{1-\sin^2(y)}} \underset{x=\sin(y)}{=} \boxed{\frac{1}{\sqrt{1-x^2}}}$$

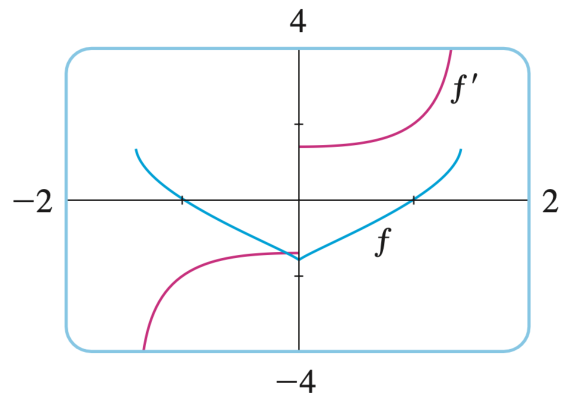
□

Example. If $f(x) = \sin^{-1}(x^2 - 1)$, find

(a) The domain of f $\to$ $[-\sqrt{2}, \sqrt{2}]$

(b) The derivative f'

(c) The domain of f' $\to$ $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$



$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.$$

(b) Let $y = \sin^{-1}(u)$, $u = x^2 - 1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot 2x$$

$$= \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x$$

$$= \frac{1}{\sqrt{1-(x^4-2x^2+1)}} \cdot 2x$$

$$= \boxed{\frac{2x}{\sqrt{2x^2 - x^4}}}$$

Example. Differentiate $y = \cos^{-1}(e^{2x})$.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}},$$

Let $y = \cos^{-1}(u)$ $\quad u = e^v \quad$ $v = 2x$

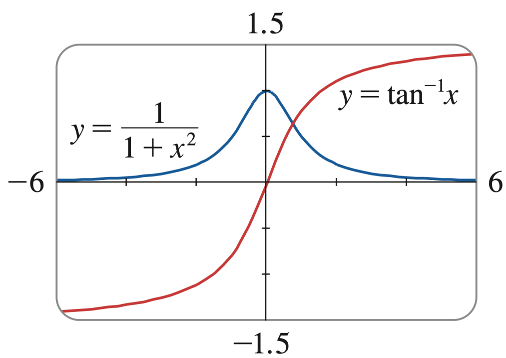
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = -\frac{1}{\sqrt{1-u^2}} \cdot e^v \cdot 2$$

$$= -\frac{1}{\sqrt{1-(e^v)^2}} \cdot e^v \cdot 2$$

$$= -\frac{1}{\sqrt{1-(e^{2x})^2}} \cdot e^{2x} \cdot 2 = \boxed{\frac{-2e^{2x}}{\sqrt{1-e^{4x}}}}$$

Theorem. Show that $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$.

Proof.



Let $y = \tan^{-1}(x)$

$\Rightarrow \tan(y) = x$

$\Rightarrow \dfrac{d}{dx}\left[\tan(y)\right] = \dfrac{d}{dx}\left[x\right]$

$\Rightarrow \sec^2(y) \cdot \dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)} = \dfrac{1}{1+\tan^2(y)} = \dfrac{1}{1+x^2}$

Trig identity
$\sec^2(y) = 1 + \tan^2(y)$

$x = \tan(y)$

□

Example. Differentiate $y = \frac{1}{\tan^{-1} x}$. $= (\arctan(x))^{-1}$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}.$$

-1 power

$y = u^{-1}$ $\quad$ $u = \arctan(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -u^{-2} \cdot \frac{1}{1+x^2}$$

$$\boxed{= -(\arctan(x))^{-2} \cdot \frac{1}{1+x^2}}$$

Example. Differentiate $y = x \arctan \sqrt{x}$.