3.6 Logarithmic Functions and Their Derivatives

$$\boxed{\frac{d}{dx}\big(\log_b(x)\big) = \frac{1}{x\ln(b)}}$$

Proof.

Let $y = \log_b(x)$

$\Rightarrow b^y = x$ [Def. of log]

$\Rightarrow \frac{d}{dx}(b^y) = \frac{d}{dx}(x)$

$\Rightarrow b^y \cdot \ln(b) \cdot \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{b^y \cdot \ln(b)} = \boxed{\frac{1}{x \cdot \ln(b)}}$

$b^y = x$

□

$$\boxed{\frac{d}{dx}\big(\ln(x)\big) = \frac{1}{x}}$$

Proof.

Since $\ln(x) = \log_e(x)$

$$\frac{d}{dx}\big(\ln(x)\big) = \frac{1}{x \cdot \ln(e)} = \frac{1}{x}$$

□

$$\frac{d}{dx}\big(\log_b(x)\big) = \frac{1}{x\ln(b)}$$

Example. Differentiate $y = \ln(x^3+1)$.

$y = \ln(u) \qquad u = x^3+1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot 3x^2$$

$$= \boxed{\frac{3x^2}{x^3+1}}$$

In general, if we have a composition of functions, the chain rule looks like:

Leibniz Notation	**Prime Notation**
$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$	$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$

Example. Find $\frac{d}{dx}\ln(\sin x)$.

$y = \ln(u) \qquad u = \sin(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \cos(x)$$

$$= \frac{\cos(x)}{\sin(x)}$$

$$= \boxed{\cot(x)}$$

Example. Differentiate $f(x) = \sqrt{\ln x}$

$y = \sqrt{u} \qquad u = \ln(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2} u^{-1/2} \cdot \frac{1}{x}$$

$$= \boxed{\frac{1}{2} (\ln(x))^{-1/2} \cdot \frac{1}{x}}$$

Example. Differentiate $f(x) = \log_{10}(2 + \sin x)$.

$y = \log_{10}(u) \qquad u = 2 + \sin(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u \cdot \ln(10)} \cdot \cos(x)$$

$$= \boxed{\frac{\cos(x)}{(2+\sin(x)) \ln(10)}}$$

Example. Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

You could start by using chain / quotient rules to get:

$$\frac{1}{\left(\frac{x+1}{\sqrt{x-2}}\right)} \cdot \frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right)$$

Is there a better way?

We can use properties of logs to <u>simplify before differentiating</u>

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(x^n) = n \cdot \log(x)$$

$$\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-2}}\right)\right) = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right]$$

$$= \boxed{\frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2}}$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

Example. Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln(y) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$\ln(y) = \ln\left(x^{3/4}\cdot\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right)$$ [Quotient Property]

$$\ln(y) = \ln\left(x^{3/4}\right) + \ln\left((x^2+1)^{1/2}\right) - \ln\left((3x+2)^5\right)$$ [Product Property]

$$\ln(y) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$ [Power Property]

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{3x+2}\cdot 3$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Example. Differentiate $y = x^{\sqrt{x}}$.

$$\ln(y) = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x} \cdot \ln(x)$$

Product Rule

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sqrt{x} \cdot \frac{d}{dx}\left(\ln(x)\right) + \ln(x) \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sqrt{x} \cdot \frac{1}{x} + \ln(x) \cdot \frac{1}{2} x^{-1/2}$$

$$\frac{dy}{dx} = y\left(\frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}}\right)$$