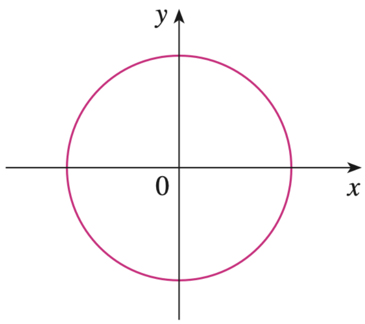
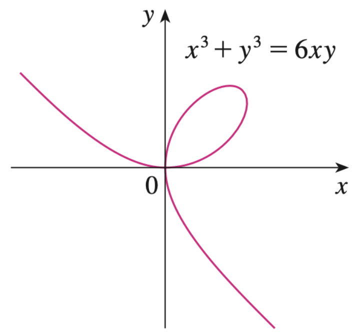
3.5 Implicit Differentiation – an application of the chain rule

Question. What is the difference between a function and a relation?



(a) $x^2 + y^2 = 25$



Functions pass the vertical line test

→ can write y in terms of x. Example: $y = x^2 + x + 1$

Relations might not pass the vertical line test

→ can't write y in terms of x. Example: $x^2 + y^2 = 25$

→ We can still graph these ⇒ We can talk about slopes

Implicit Differentiation: When y is not explicitly written as a function of x, we can still find the slope $\frac{dy}{dx}$ by differentiating *both sides with respect to* x and using the Chain Rule. We view y as an *unknown function* $y(x)$. key idea of implicit differentiation

1. **Start from the relation.** Write the given equation relating the variables x and y.
2. **Differentiate both sides with respect to x.** Apply derivative rules as needed. Every time you differentiate a term involving y, multiply by a factor $y' = \frac{dy}{dx}$ via the Chain Rule.
3. **Collect y'-terms.** Move all terms containing y' to one side and all other terms to the opposite side.
4. **Factor and solve for y'.** Factor out y' and isolate it to obtain an explicit formula $y' = \frac{dy}{dx}$ in terms of x and y.

Example.

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

(a) $x^2 + y^2 = 25$

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$\downarrow$ chain rule

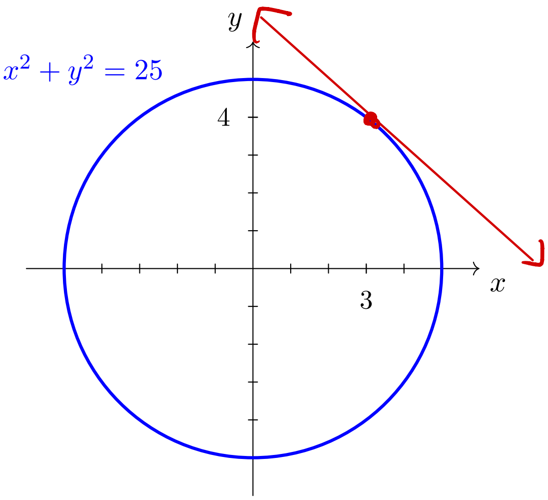
$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \boxed{\frac{-x}{y}}$$

(b) At $(3,4)$ the slope is $-\frac{3}{4}$

$$\boxed{y - 4 = -\frac{3}{4}(x-3)}$$



Big idea of implicit differentiation:

$$\frac{d}{dy}(y^2) = 2y$$

BUT

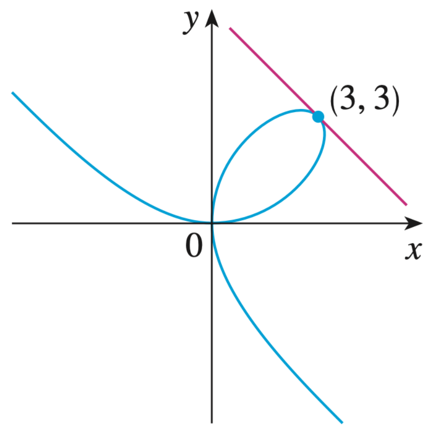
Leibniz Notation $\quad \dfrac{d}{dx}(y^2) = 2y \cdot \dfrac{dy}{dx}$

Prime Notation $\quad \dfrac{d}{dx}\left[(y(x))^2\right] = 2 \cdot y(x) \cdot y'(x)$

Example.

(a) Find y' if $x^3 + y^3 = 6xy$

(b) Find an equation of the tangent line to the folium of Descartes $x^3+y^3 = 6xy$ at the point $(3, 3)$



(a) $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}(6xy)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6 \cdot \frac{d}{dx}(xy)$$

chain rule   product rule

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6 \cdot \left[x \cdot \frac{dy}{dx} + y \cdot 1\right]$$

Remember: y is $y(x)$

$$\frac{d}{dx}\left[(y(x))^3\right] = 3(y(x))^2 \cdot y'(x)$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6x \cdot \frac{dy}{dx} + 6y$$

$$3y^2 \cdot \frac{dy}{dx} - 6x \cdot \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 6x\right) = 6y - 3x^2$$

solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$$

(b) At $(3,3)$, $\frac{dy}{dx} = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = \frac{6 - 9}{9 - 6} = \boxed{-1}$

$$\boxed{y - 3 = -1(x - 3)}$$

Example. Find y' if $\sin(x+y) = y^2\cos(x)$.

$$\frac{d}{dx}\left(\sin(x+y)\right) = \frac{d}{dx}\left(y^2\cos(x)\right)$$

chain rule   product rule

$$\cos(x+y)\cdot\frac{d}{dx}(x+y) = y^2\cdot\frac{d}{dx}\left(\cos(x)\right) + \cos(x)\cdot\frac{d}{dx}(y^2)$$

chain rule

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = y^2\cdot\left(-\sin(x)\right) + \cos(x)\cdot 2y\cdot\frac{dy}{dx}$$

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2\sin(x) + 2y\cdot\cos(x)\cdot\frac{dy}{dx}$$

$$\cos(x+y)\cdot\frac{dy}{dx} - 2y\cdot\cos(x)\cdot\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$$

$$\frac{dy}{dx}\left(\cos(x+y) - 2y\cos(x)\right) = -y^2\sin(x) - \cos(x+y)$$

Solve for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{-y^2\sin(x) - \cos(x+y)}{\cos(x+y) - 2y\cos(x)}}$$

Example. Find y' if $x^4 + y^4 = 16$.

$$\frac{d}{dx}\left(x^4+y^4\right) = \frac{d}{dx}(16)$$

* you can use y' notation instead of $\frac{dy}{dx}$ if you prefer

$$\frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(y^4\right) = \frac{d}{dx}(16)$$

$$4x^3 + 4y^3 \cdot y' = 0$$

$$4y^3 \cdot y' = -4x^3$$

$$y' = \frac{-4x^3}{4y^3} = \boxed{\frac{-x^3}{y^3}}$$