3.4 The Chain Rule

Theorem. If $u = g(x)$ is differentiable at x and $y = f(u)$ is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x, and F' is given by the product:

The chain rule

Leibniz Notation	**Prime Notation**
$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$	$F'(x) = f'\big(g(x)\big)\, g'(x)$

Example. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

① Prime notation:

$f(u) = \sqrt{u}$ $\qquad g(x) = x^2+1$

$f'(u) = \frac{1}{2}u^{-1/2}$ $\qquad g'(x) = 2x$

Check: $f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$ ✓

$$F'(x) = f'\Big(g(x)\Big) \cdot g'(x) \quad \text{[chain rule]}$$
$$= f'(x^2+1) \cdot 2x$$
$$= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \boxed{\frac{x}{\sqrt{x^2+1}}}$$

② Leibniz notation:

$y = \sqrt{u}$ $\qquad u = x^2+1$

Check: $y = \sqrt{x^2+1}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{[chain rule]}$$
$$= \frac{1}{2}u^{-1/2} \cdot 2x$$
$$= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Example. Differentiate $y = \sin(x^2)$.

① Prime notation:

Check: $f(g(x)) = f(x^2)$
$= \sin(x^2)$ ✓

$f(u) = \sin(u)$ $\quad$ $g(x) = x^2$

$f'(u) = \cos(u)$ $\quad$ $g'(x) = 2x$

$F'(x) = f'(g(x)) \cdot g'(x)$ $\quad$ [Chain rule]

$= f'(x^2) \cdot 2x$

$= \boxed{\cos(x^2) \cdot 2x}$

② Leibniz notation:

$y = \sin(u)$ $\quad$ $u = x^2$ $\quad$ check: $y = \sin(x^2)$ ✓

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\quad$ [chain rule]

$= \cos(u) \cdot 2x$

$= \boxed{\cos(x^2) \cdot 2x}$

Example. Differentiate $y = \sin^2(x)$. $= [\sin(x)]^2$

① Prime notation

$f(u) = u^2$ $\quad$ $g(x) = \sin(x)$ $\quad$ check: $f(g(x)) = f(\sin(x))$

$f'(u) = 2u$ $\quad$ $g'(x) = \cos(x)$ $\quad$ $= [\sin(x)]^2$

$$F'(x) = f'(g(x)) \cdot g'(x) \quad \text{[chain rule]}$$

$$= f'(\sin(x)) \cdot \cos(x)$$

$$= \boxed{2\sin(x)\cos(x)}$$

② Leibniz notation

$y = u^2$ $\quad$ $u = \sin(x)$ $\quad$ Check: $y = [\sin(x)]^2$ ✓

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{[chain rule]}$$

$$= 2u \cdot \cos(x)$$

$$= \boxed{2\sin(x)\cos(x)}$$

Example. Differentiate $y = (x^3 - 1)^{100}$

① Prime notation:

$f(u) = u^{100}$ $\qquad$ $g(x) = x^3 - 1$

$f'(u) = 100u^{99}$ $\qquad$ $g'(x) = 3x^2$

Check: $f(g(x)) = f(x^3-1)$

$= (x^3-1)^{100}$ ✓

$$f'(g(x)) \cdot g'(x) = f'(x^3-1) \cdot 3x^2$$
$$= \boxed{100(x^3-1)^{99} \cdot 3x^2}$$

② Leibniz notation:

$y = u^{100}$ $\qquad$ $u = x^3 - 1$

Check: $y = (x^3-1)^{100}$ ✓

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 100u^{99} \cdot 3x^2$$
$$= \boxed{100(x^3-1)^{99} \cdot 3x^2}$$

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Example. Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$. $= (x^2+x+1)^{-1/3}$

① Prime notation:

$h(u) = u^{-1/3}$ $g(x) = x^2+x+1$

$h'(u) = -\frac{1}{3}u^{-4/3}$ $g'(x) = 2x+1$

$h'(g(x))\cdot g'(x) = h'(x^2+x+1)\cdot(2x+1)$

$= \boxed{-\frac{1}{3}(x^2+x+1)^{-4/3}\cdot(2x+1)}$

② Leibniz Notation:

$y = u^{-1/3}$ $u = x^2+x+1$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= -\frac{1}{3}u^{-4/3}\cdot(2x+1)$

$= \boxed{-\frac{1}{3}(x^2+x+1)^{-4/3}\cdot(2x+1)}$

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Example. Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

Remark: In general, $\frac{d}{dx}\left[h(x)\right]^n = n\left[h(x)\right]^{n-1} \cdot h'(x)$

By our observation,

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}$$

$$= \frac{9(t-2)^8}{(2t+1)^8} \cdot \frac{5}{(2t+1)^2}$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example. Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$

One line:

$$\boxed{\frac{dy}{dx} = (2x+1)^5 \cdot 4(x^3-x+1)^3 \cdot (3x^2-1) + (x^3-x+1)^4 \cdot 5(2x+1)^4 \cdot 2}$$

$y = u^5 \cdot v^4$ $\quad$ $u = 2x+1, \quad v = x^3 - x + 1$

$y = p \cdot q$ $\quad$ $p = u^5 \quad , \quad q = v^4$

$$\frac{dy}{dx} = p \cdot \frac{dq}{dx} + q \cdot \frac{dp}{dx} \qquad \text{[product rule]}$$

$$= p \cdot \frac{dq}{dv} \cdot \frac{dv}{dx} + q \cdot \frac{dp}{du} \cdot \frac{du}{dx} \qquad \text{[chain rule]}$$

$$= p \cdot 4v^3 \cdot (3x^2-1) + q \cdot 5u^4 \cdot 2$$

$$= u^5 \cdot 4v^3 \cdot (3x^2-1) + v^4 \cdot 5u^4 \cdot 2 \qquad \text{[replace } p, q\text{]}$$

$$= (2x+1)^5 \cdot 4(x^3-x+1)^3 \cdot (3x^2-1) + (x^3-x+1)^4 \cdot 5(2x+1)^4 \cdot 2 \qquad \text{[replace } u, v\text{]}$$

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Example. Differentiate $y = e^{\sin x}$

① Prime notation:

Check: $f(g(x)) = f(\sin(x))$
$= e^{\sin(x)}$

$f(u) = e^u \qquad g(x) = \sin(x)$

$f'(u) = e^u \qquad g'(x) = \cos(x)$

$$f'(g(x)) \cdot g'(x) = f'(\sin(x)) \cdot \cos(x)$$

$$= e^{\sin(x)} \cdot \cos(x)$$

② Leibniz Notation:

$y = e^u \qquad u = \sin(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u \cdot \cos(x)$$

$$= e^{\sin(x)} \cdot \cos(x)$$

Example. Differentiate $p(x) = \sin(\cos(\tan x))$.

① Leibniz Notation:

$$y = \sin(u) \qquad u = \cos(v) \qquad v = \tan(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \qquad \text{[chain rule]}$$

$$= \cos(u) \cdot -\sin(v) \cdot \sec^2(x) \qquad \text{[take derivatives]}$$

$$= \cos(\cos(v)) \cdot -\sin(v) \cdot \sec^2(x) \qquad \text{[replace } u\text{]}$$

$$= \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2(x) \qquad \text{[replace } v\text{]}$$

② Prime notation:

$$f(u) = \sin(u) \qquad g(v) = \cos(v) \qquad h(x) = \tan(x)$$

$$f'(u) = \cos(u) \qquad g'(v) = -\sin(v) \qquad h'(x) = \sec^2(x)$$

$$\frac{d}{dx}\left[f(g(h(x)))\right] = f'(g(h(x))) \cdot \frac{d}{dx}\left[g(h(x))\right]$$

$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

$$= \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2(x)$$

Example. Differentiate $y = e^{\sec 3\theta}$.

$\frac{d}{dx}\left[\sec(x)\right] = \sec(x) \cdot \tan(x)$

$y = e^u$ $\quad u = \sec(v)$ $\quad v = 3\theta$

$$\begin{array}{c} y \\ | \\ u \\ | \\ v \\ | \\ \theta \end{array}$$

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{d\theta}$$

$$= e^u \cdot \sec(v)\tan(v) \cdot 3$$

$$= e^{\sec(v)} \cdot \sec(v)\tan(v) \cdot 3$$

$$= 3e^{\sec(3\theta)} \cdot \sec(3\theta)\tan(3\theta)$$

Theorem. Let $a > 0$ with $a \neq 1$. Then

$$\frac{d}{dx}(a^x) \;=\; a^x \ln a.$$

Proof.

Since $a = e^{\ln(a)}$, $a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a) \cdot x}$

Let $y = e^u$, $u = \ln(a) \cdot x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \ln(a) = e^{\ln(a) \cdot x} \cdot \ln(a) = \boxed{a^x \cdot \ln(a)}$$

□

Example. Differentiate $g(x) = 2^x$

$$g'(x) = 2^x \cdot \ln(2)$$

Example. Differentiate $g(x) = 5^{x^2}$

$y = 5^u \qquad u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5^u \cdot \ln(5) \cdot 2x$$

$$= 5^{x^2} \cdot \ln(5) \cdot 2x$$