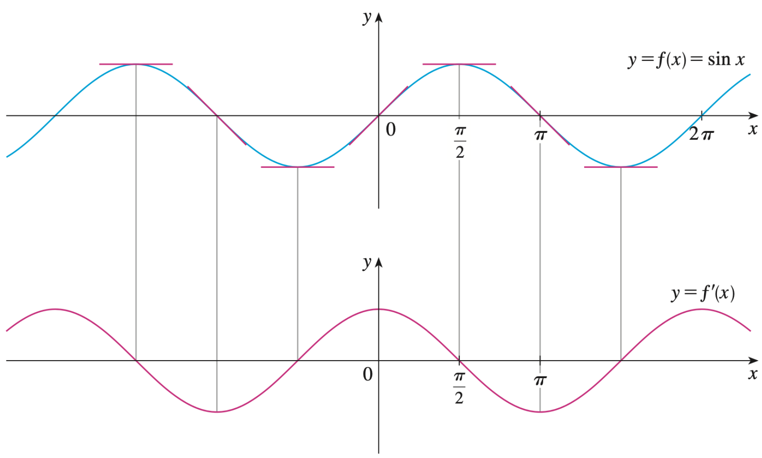
3.3 Derivatives of Trigonometric Functions

Theorem.

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x\ \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\ \csc^2 x \qquad \frac{d}{dx}(\csc x) = -\ \csc x\ \cot x$$

Proof. Let $f(x)$ be $\sin x$. We will show that $f'(x) = \cos x$.



$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\qquad\qquad\qquad\qquad}{h}$$

$$= \lim_{h\to 0}\frac{\qquad\qquad\qquad\qquad\qquad\qquad}{h}$$

$$= \lim_{h\to 0}\left[\frac{\qquad\qquad\qquad}{h} + \frac{\qquad\qquad\qquad}{h}\right]$$

$$= \lim_{h\to 0}\left[\qquad\cdot\left(\frac{\qquad\qquad}{h}\right) + \qquad\cdot\left(\frac{\qquad\qquad}{h}\right)\right]$$

$$= \qquad\cdot\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \qquad\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= \qquad\cdot\qquad+\qquad\cdot\qquad=$$

□

Example. Differentiate $y = x^2 \sin(x)$.

Proof. Let $f(x)$ be $\tan(x)$. We will show that $f'(x) = \sec^2(x)$.

$$\frac{d}{dx}\big(\tan x\big) = \frac{d}{dx}\Big(\frac{\qquad}{\qquad}\Big)$$

$$= \frac{(\qquad\qquad)\cdot\frac{d}{dx}(\qquad\qquad) - (\qquad\qquad)\cdot\frac{d}{dx}(\qquad\qquad)}{(\cos x)^2}$$

$$= \frac{\qquad\qquad\qquad\qquad}{\cos^2 x}$$

$$= \frac{\qquad\qquad\qquad}{\cos^2 x}$$

$$= \frac{\qquad}{\cos^2 x}$$

$$= \sec^2 x.$$

$\square$

Example. For what values of x does the graph of $f(x) = \dfrac{\sec(x)}{1+\tan(x)}$ have a horizontal tangent?

$$f'(x) = \frac{\left(\qquad\qquad\qquad\qquad \right) \frac{d}{dx}\left(\qquad\qquad\qquad\qquad \right) - \left(\qquad\qquad\qquad\qquad \right) \frac{d}{dx}\left(\qquad\qquad\qquad\qquad \right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\left(1+\tan x\right) \cdot \qquad\qquad\qquad\qquad\qquad\qquad - \ \sec x \cdot \qquad\qquad\qquad\qquad\qquad\qquad}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x \left(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x \left(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x \left(\tan x + \tan^2 x - (\qquad\qquad\qquad\qquad) \right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\qquad\qquad\qquad\qquad\qquad\qquad}{\left(1+\tan x\right)^2}.$$

Example. Find the 27th derivative of $\cos(x)$.

Example. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Its position at time t is $s = f(t) = 4\cos t$. Find the velocity and acceleration at time t and use them to analyze the motion of the object.