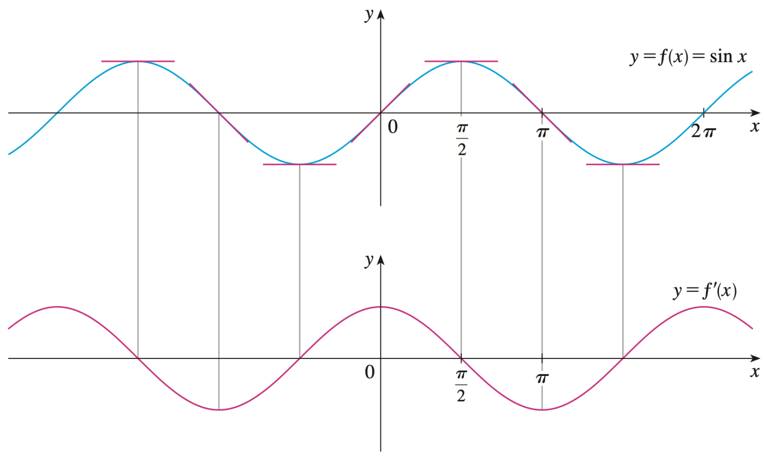
3.3 Derivatives of Trigonometric Functions

Theorem.

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x\, \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x\, \cot x$$

Proof. Let $f(x)$ be $\sin x$. We will show that $f'(x) = \cos x$.



$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

[Trig identity]

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \right]$$

$$= \lim_{h\to 0} \left[\sin(x) \cdot \left(\frac{\cos(h) - 1}{h} \right) + \cos(x) \cdot \left(\frac{\sin(h)}{h} \right) \right]$$

$$= \sin(x) \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \cos(x) \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)$$

Algebra

Squeeze Thm

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \boxed{\cos(x)}$$

□

Example. Differentiate $y = x^2 \sin(x)$.

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\sin(x)) + \sin(x) \cdot \frac{d}{dx}(x^2)$$

$$= x^2 \cdot \cos(x) + \sin(x) \cdot 2x$$

Proof. Let $f(x)$ be $\tan(x)$. We will show that $f'(x) = \sec^2(x)$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{(\cos(x)) \cdot \frac{d}{dx}(\sin(x)) - (\sin(x)) \cdot \frac{d}{dx}(\cos(x))}{(\cos x)^2}$$

$$= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2 x}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

□

Example. For what values of x does the graph of $f(x) = \dfrac{\sec(x)}{1+\tan(x)}$ have a horizontal tangent?

$$f'(x) = \frac{\left(1+\tan(x) \right) \frac{d}{dx}\left(\sec(x) \right) - \left(\sec(x) \right) \frac{d}{dx}\left(1+\tan(x) \right)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\cdot \sec(x)\tan(x) - \sec x \cdot \sec^2(x)}{(1+\tan x)^2}$$

$$= \frac{\sec x\left((1+\tan(x))\cdot \tan(x) - \sec^2(x) \right)}{(1+\tan x)^2}$$

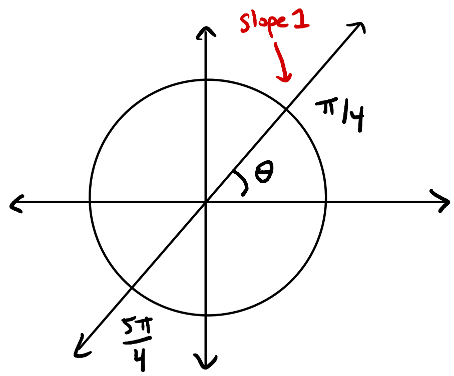
$$= \frac{\sec x\left(\tan(x) + \tan^2(x) - \sec^2(x) \right)}{(1+\tan x)^2}$$

$\sec^2(x) = 1+\tan^2(x)$

$$= \frac{\sec x\left(\tan x + \tan^2 x - \left(1+\tan^2(x) \right)\right)}{(1+\tan x)^2}$$

$$= \frac{\sec(x)\left(\tan(x) - 1\right)}{(1+\tan x)^2}.$$

tan(θ) is the slope of the line corresponding to θ



This is 0 when the numerator is 0 (and the denominator is nonzero)

- $\sec(x)$ is never 0
- $\tan(x) - 1 = 0$ when $\tan(x) = 1$

$$\boxed{x = \frac{\pi}{4} + \pi n}$$

Example. Find the 27th derivative of $\cos(x)$.

Example. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Its position at time t is $s = f(t) = 4\cos t$. Find the velocity and acceleration at time t and use them to analyze the motion of the object.