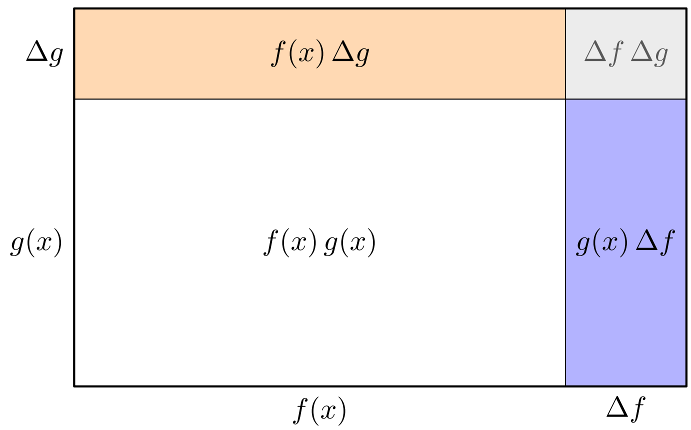
3.2 The Product and Quotient Rules

Theorem (Product Rule)**.** If f and g are both differentiable, then their derivatives satisfy:

Leibniz Notation	**Prime Notation**
$\dfrac{d}{dx}\big[f(x)g(x)\big] \;=\; f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$	$(fg)'(x) \;=\; f(x)g'(x) + g(x)f'(x)$

Proof #1: Using Geometry.



$\square$

Proof #2: Using the Limit Definition of the Derivative.

Let $A(x) = f(x)g(x)$. Then

$$\begin{aligned}
A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(f(x+h)\,\frac{g(x+h) - g(x)}{h} + g(x)\,\frac{f(x+h) - f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)\,g'(x) + g(x)\,f'(x).
\end{aligned}$$

Note: $f(x+h) \to f(x)$ as $h \to 0$, since f is differentiable, and therefore continuous, at x.

$\square$

Example. If $f(x) = xe^x$, find $f'(x)$. What is the nth derivative $f^{(n)}(x)$?

Example. Differentiate the function $f(t) = \sqrt{t} \cdot (5 + 8t)$.

Example. If $f(x) = \sqrt{x} \cdot g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Theorem (Quotient Rule)**.** If f and g are differentiable at x and $g(x) \neq 0$, then their derivatives satisfy:

Leibniz Notation

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \;=\; \frac{g(x)\,\frac{d}{dx}f(x) \;-\; f(x)\,\frac{d}{dx}g(x)}{\big(g(x)\big)^2}$$

Prime Notation

$$\left(\frac{f}{g}\right)'(x) \;=\; \frac{f'(x)\,g(x) \;-\; f(x)\,g'(x)}{\big(g(x)\big)^2}$$

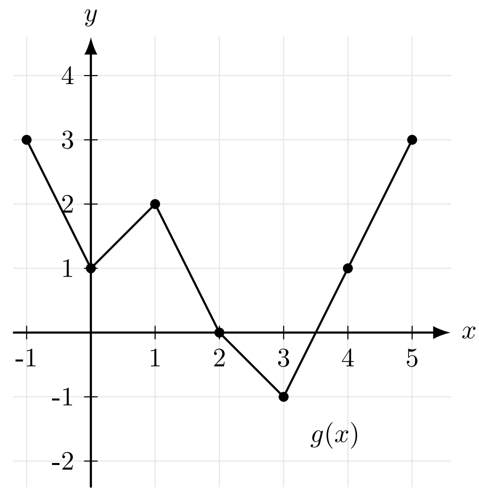
Example. Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. What is y'?

Example. Find an equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

Example. Should we use the quotient rule to differentiate $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$?

Example. A table of values for $f(x)$ and $f'(x)$ and a graph of a piecewise linear function $g(x)$ are shown below.

x	$f(x)$	$f'(x)$
-1	5	-1
0	-2	4
1	0	3
2	7	-2
3	9	5
4	3	1



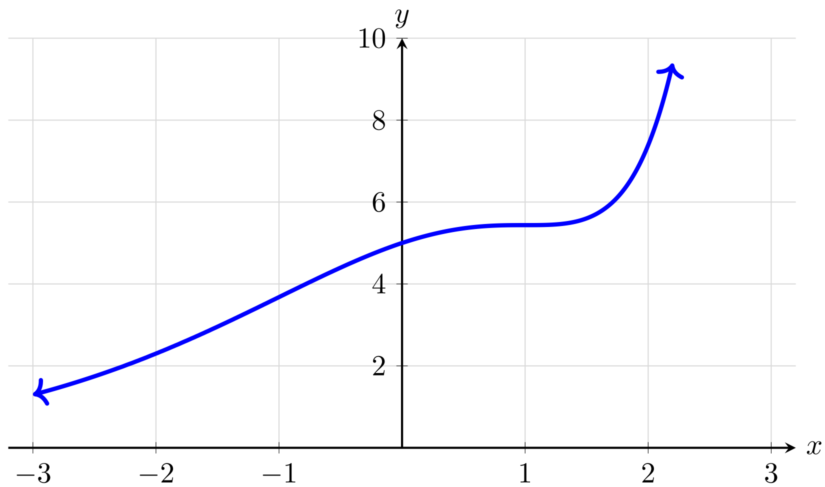
(a) Let $p(x) = f(x)\,g(x)$. Find $p'(1)$ and $p'(2)$.

(b) Let $q(x) = \dfrac{g(x)}{f(x)+1}$. Find $q'(4)$.

Example. Use the function $f(x) = (x^2 - 4x + 5)e^x$ to answer:

(a) On which interval(s) is $f(x)$ increasing?

(b) On which interval(s) is $f(x)$ concave upward?



Example. Compute $f'(1)$, where $f(x) = \dfrac{4x^2}{\sqrt{x}e^x}$.