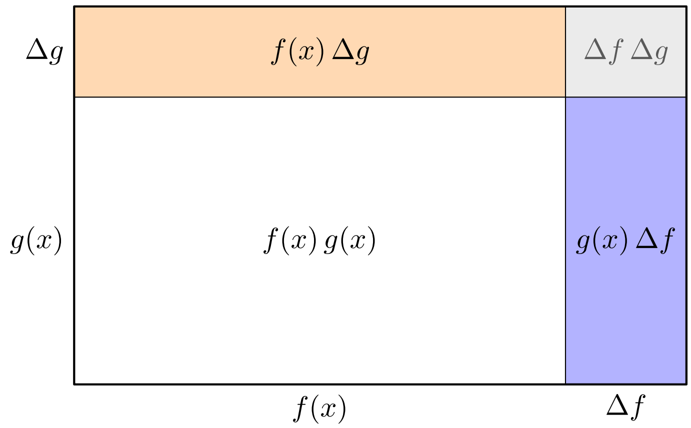
3.2 The Product and Quotient Rules

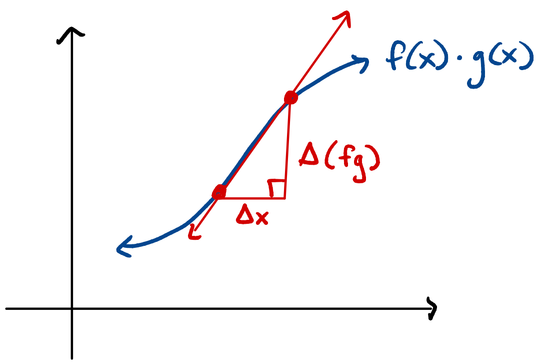
Theorem (Product Rule)**.** If f and g are both differentiable, then their derivatives satisfy:

Leibniz Notation	**Prime Notation**
$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$	$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$

Proof #1: Using Geometry.

Let Δx be a small change in x. This produces a small change in $f(x)$, $g(x)$, and the product $f(x) \cdot g(x)$





$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta(fg)}{\Delta x} = \frac{f \cdot \Delta g}{\Delta x} + \frac{g \cdot \Delta f}{\Delta x} + \frac{\Delta f \Delta g}{\Delta x}$$

As $\Delta x \to 0$: $\Delta f \to 0$, $\Delta g \to 0$, $\frac{\Delta g}{\Delta x} \to \frac{dg}{dx}$, $\frac{\Delta f}{\Delta x} \to \frac{df}{dx}$

$$\frac{d(fg)}{dx} = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx} + 0 \cdot \frac{dg}{dx}$$

$\square$

Proof #2: Using the Limit Definition of the Derivative.

Let $A(x) = f(x)g(x)$. Then

$$
\begin{aligned}
A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(f(x+h)\,\frac{g(x+h)-g(x)}{h} + g(x)\,\frac{f(x+h)-f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= f(x)\,g'(x) + g(x)\,f'(x).
\end{aligned}
$$

Note: $f(x+h) \to f(x)$ as $h \to 0$, since f is differentiable, and therefore continuous, at x.

$\square$

Example. If $f(x) = xe^x$, find $f'(x)$. What is the nth derivative $f^{(n)}(x)$?

Product Rule: $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot \frac{d}{dx}[g(x)] + g(x)\cdot \frac{d}{dx}[f(x)]$

$$
\begin{aligned}
f'(x) = \frac{d}{dx}(xe^x) &= x\cdot \frac{d}{dx}(e^x) + e^x\cdot \frac{d}{dx}(x) \\
&= x\cdot e^x + e^x \\
&= (x+1)\cdot e^x
\end{aligned}
$$

$$
\begin{aligned}
f''(x) = \frac{d}{dx}\left((x+1)e^x\right) &= (x+1)\frac{d}{dx}(e^x) + e^x\cdot \frac{d}{dx}(x+1) \\
&= (x+1)\cdot e^x + e^x \\
&= (x+2)e^x
\end{aligned}
$$

By repeated applications of the product rule, $f^{(n)}(x) = (x+n)e^x$

Example. Differentiate the function $f(t) = \sqrt{t} \cdot (5 + 8t)$.

① Product Rule: $f'(t) = \sqrt{t} \cdot \frac{d}{dt}(5+8t) + (5+8t) \cdot \frac{d}{dt}(\sqrt{t})$ $\quad \sqrt{t} = t^{1/2}$

$$= \sqrt{t} \cdot 8 + (5+8t) \cdot \tfrac{1}{2} t^{-1/2}$$

$$= \boxed{8\sqrt{t} + \frac{5+8t}{2\sqrt{t}}}$$

same!

② Simplify first, then Power Rule:

$$f(t) = 5t^{1/2} + 8t^{3/2}$$

$$f'(t) = \tfrac{1}{2} \cdot 5t^{-1/2} + \tfrac{3}{2} \cdot 8t^{1/2} = \boxed{\frac{5}{2\sqrt{t}} + \frac{24\sqrt{t}}{2}}$$

Example. If $f(x) = \sqrt{x} \cdot g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Use the product rule:

$$f'(x) = \sqrt{x} \cdot g'(x) + g(x) \cdot \frac{d}{dx}(\sqrt{x})$$ $\quad x^{1/2}$

$$= \sqrt{x} \cdot g'(x) + g(x) \cdot \tfrac{1}{2} x^{-1/2}$$

$$f'(4) = \sqrt{4} \cdot g'(4) + g(4) \cdot \tfrac{1}{2} \cdot (4)^{-1/2}$$

$$= 2 \cdot 3 + 2 \cdot \tfrac{1}{2} \cdot \frac{1}{\sqrt{4}}$$

$$= 6 + \tfrac{1}{2}$$

$$= \boxed{6.5}$$

"Low d high minus high d low ... all over what is squared below"

Theorem (Quotient Rule)**.** If f and g are differentiable at x and $g(x) \neq 0$, then their derivatives satisfy:

Leibniz Notation

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\frac{d}{dx}f(x) \;-\; f(x)\,\frac{d}{dx}g(x)}{\big(g(x)\big)^2}$$

Prime Notation

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) \;-\; f(x)\,g'(x)}{\big(g(x)\big)^2}$$

Example. Let $y = \dfrac{x^2+x-2}{x^3+6}$. What is y'?

$$y' = \frac{(x^3+6)\cdot\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\cdot\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)\cdot(2x+1) - (x^2+x-2)\cdot(3x^2)}{(x^3+6)^2}$$

[Stopping here is ok]

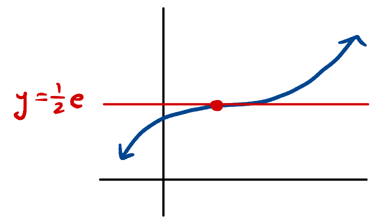
$$= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \boxed{\frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}}$$

Example. Find an equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

$$y' = \frac{(1+x^2)\cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\cdot e^x - e^x \cdot (2x)}{(1+x^2)^2}$$



$$y'(1) = \frac{(1+1)\cdot e^1 - e^1 \cdot 2}{2^2} = \frac{2e - 2e}{4} = 0$$

Horizontal tangent line! $\boxed{y = \frac{1}{2}e}$

Example. Should we use the quotient rule to differentiate $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$?

We could use the quotient rule, but we can simplify $F(x)$ first.

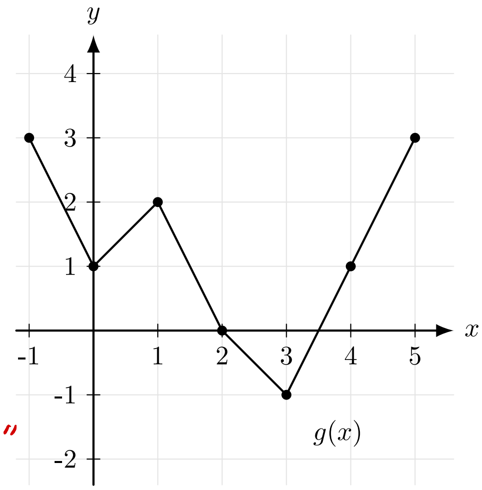
$$F(x) = \frac{3x^2}{x} + \frac{2x^{1/2}}{x}$$

$$= 3x + 2x^{-1/2}$$

$$\boxed{F'(x) = 3 - x^{-3/2}}$$

Example. A table of values for $f(x)$ and $f'(x)$ and a graph of a piecewise linear function $g(x)$ are shown below.

x	$f(x)$	$f'(x)$
-1	5	-1
0	-2	4
1	0	3
2	7	-2
3	9	5
4	3	1



"Wacky derivative"

(a) Let $p(x) = f(x)\,g(x)$. Find $p'(1)$ and $p'(2)$.

(b) Let $q(x) = \dfrac{g(x)}{f(x)+1}$. Find $q'(4)$.

ⓐ $p'(1) = f'(1)g(1) + f(1)\underbrace{g'(1)}_{\text{DNE}}$

From the left: $3 \cdot 2 + 0 \cdot 1 = 6$

From the right: $3 \cdot 2 + 0 \cdot -2 = 6$

$\boxed{6}$

$p'(2) = f'(2)g(2) + f(2)\underbrace{g'(2)}_{\text{DNE}}$

From the left: $-2 \cdot 0 + 7 \cdot (-2) = -14$

From the right: $-2 \cdot 0 + 7 \cdot (-1) = -7$

$\boxed{\text{DNE}}$

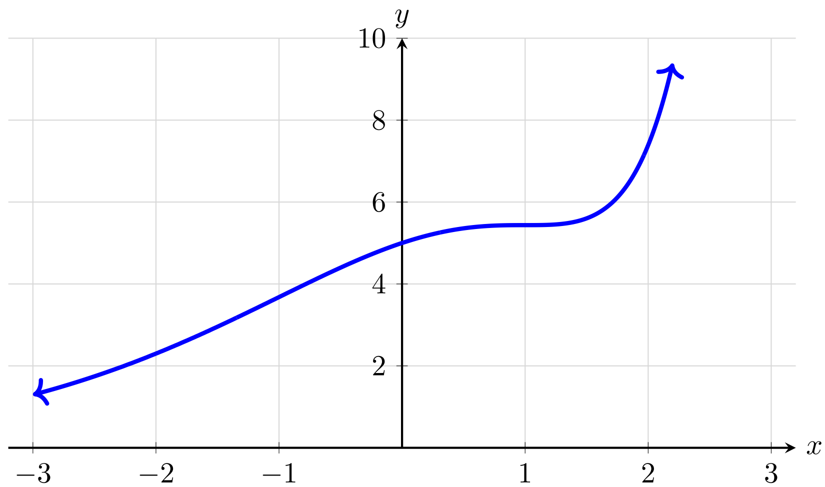
ⓑ $$q'(x) = \frac{(f(x)+1) \cdot \frac{d}{dx}(g(x)) - g(x) \cdot \frac{d}{dx}(f(x)+1)}{(f(x)+1)^2} = \frac{(f(x)+1) \cdot g'(x) - g(x) \cdot f'(x)}{(f(x)+1)^2}$$

$$q'(4) = \frac{(f(4)+1) \cdot g'(4) - g(4) \cdot f'(4)}{(f(4)+1)^2} = \frac{4 \cdot 2 - 1 \cdot 1}{16} = \boxed{\frac{7}{16}}$$

Example. Use the function $f(x) = (x^2 - 4x + 5)e^x$ to answer:

(a) On which interval(s) is $f(x)$ increasing? $\longrightarrow$ increasing on (a, b): $f(a) < f(b)$ for $a < b$

(b) On which interval(s) is $f(x)$ concave upward?

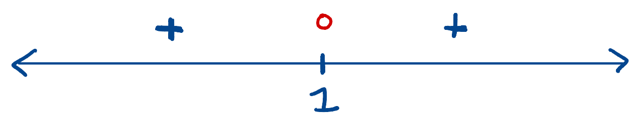


$$f'(x) = (x^2 - 4x + 5) \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x^2 - 4x + 5)$$

$$= (x^2 - 4x + 5) \cdot e^x + e^x \cdot (2x - 4)$$

$$= (x^2 - 2x + 1) \cdot e^x$$

$$= (x - 1)^2 \cdot e^x$$



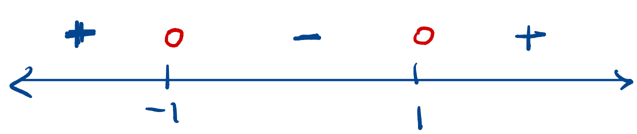
$\boxed{(-\infty, \infty)}$ (it's increasing at $x = 1$ too!)

$$f''(x) = (x^2 - 2x + 1) \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2 - 2x + 1) \cdot e^x$$

$$= (x^2 - 2x + 1) \cdot e^x + (2x - 2) \cdot e^x$$

$$= (x^2 - 1)e^x$$

$$= (x + 1)(x - 1)e^x$$



$\boxed{(-\infty, -1) \cup (1, \infty)}$

Example. Compute $f'(1)$, where $f(x) = \dfrac{4x^2}{\sqrt{x}e^x}$.

① Product & Quotient Rule:

$$f'(x) = \frac{(\sqrt{x}\cdot e^x)\frac{d}{dx}(4x^2) - 4x^2\frac{d}{dx}(\sqrt{x}\,e^x)}{(\sqrt{x}\,e^x)^2}$$

$$= \frac{(\sqrt{x}\,e^x)\cdot 8x - 4x^2\cdot\left[\sqrt{x}\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\sqrt{x})\right]}{xe^{2x}}$$

$$= \frac{\sqrt{x}\cdot e^x\cdot 8x - 4x^2\left(\sqrt{x}\cdot e^x + e^x\cdot\frac{1}{2}x^{-1/2}\right)}{xe^{2x}}$$

$$f'(1) = \frac{1\cdot e\cdot 8 - 4\left(1\cdot e + e\cdot\frac{1}{2}\cdot 1\right)}{1\cdot e^2}$$

Plug in $x=1$ ASAP

$$= \frac{8e - 4e - 2e}{e^2} = \frac{2e}{e^2} = \frac{2}{e}$$

② Simplify first:

$$f(x) = \frac{4x^{3/2}}{e^x}$$

$$f'(x) = \frac{e^x\cdot\frac{d}{dx}(4x^{3/2}) - 4x^{3/2}\cdot\frac{d}{dx}(e^x)}{(e^x)^2} = \frac{e^x\cdot 6x^{1/2} - 4x^{3/2}\cdot e^x}{e^{2x}}$$

$$f'(1) = \frac{e\cdot 6\cdot 1 - 4\cdot 1\cdot e^1}{e^2} = \frac{2e}{e^2} = \frac{2}{e}$$