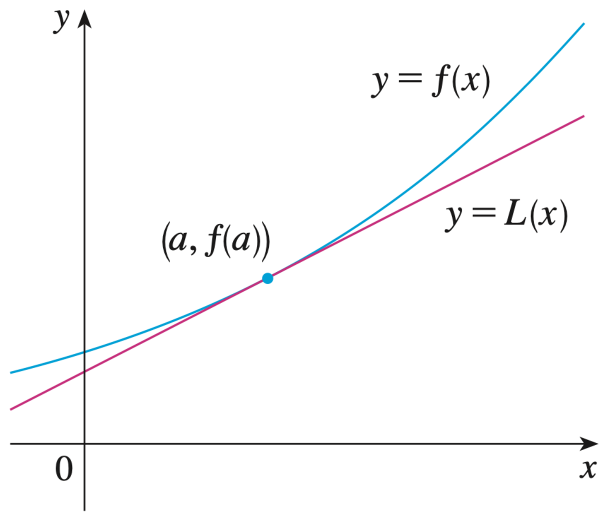
3.10 Linear Approximation

Question. We can use the tangent line at $(a, f(a))$ to approximate $y = f(x)$. What is the equation of this tangent line?

Point-slope: $y - y_1 = m(x - x_1)$

$$y - f(a) = f'(a)(x - a)$$

$$y = f'(a)(x-a) + f(a)$$

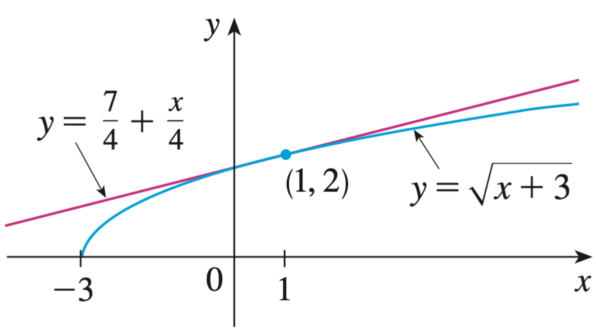


$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$

↑ Equation of the tangent line at $(a, f(a))$

Sometimes called the "Linearization" of $f(x)$

Example. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?



① $L(x) = f'(1)(x-1) + f(1)$

$f'(x) = \frac{1}{2}(x+3)^{-1/2}$

$f'(1) = \frac{1}{2}(4)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4}$

$f(1) = \sqrt{1+3} = 2$

$$\boxed{L(x) = \frac{1}{4}(x-1) + 2 = \frac{x}{4} + \frac{7}{4}}$$

② $\sqrt{3.98} = f(0.98) \approx L(0.98) = \frac{0.98}{4} + \frac{7}{4} = \boxed{1.995}$

$\sqrt{4.05} = f(1.05) \approx L(1.05) = \frac{1.05}{4} + \frac{7}{4} = \boxed{2.0125}$

③ These are overestimates because $L(x)$ lies above the curve.

(because $f(x)$ is concave down)

Example. Use a linear approximation to estimate $e^{-0.015}$.

Example. Use a linear approximation to estimate $(2.001)^5$.