3.1 Derivatives of Polynomials and Exponential Functions

Definition. With respect to the variable x, the symbol

$$\frac{d}{dx}$$

means "*take the derivative with respect to* x." It is an *operator* that takes a function f as input and produces the derivative function as output. The functions

$$\underbrace{\frac{d}{dx}\big[f(x)\big]}_{\text{operator on } f} \;=\; \underbrace{f'(x)}_{\text{Lagrange}} \;=\; \underbrace{\frac{df}{dx}}_{\text{Leibniz}}$$

all denote the same derivative function. If $y = f(x)$, then

$$\frac{dy}{dx} \;=\; f'(x) \;=\; \frac{d}{dx}\big[f(x)\big].$$

In contexts where the independent variable is *time* t, one often uses Newton's dot notation:

$$\dot{y} \;=\; \frac{dy}{dt}, \qquad \ddot{y} \;=\; \frac{d^2y}{dt^2}.$$

The second and nth derivatives can be written as

$$f''(x) \;=\; \frac{d^2 f}{dx^2} \;=\; \frac{d^2}{dx^2}\big[f(x)\big] \qquad f^{(n)}(x) \;=\; \frac{d^n f}{dx^n} \;=\; \frac{d^n}{dx^n}\big[f(x)\big]$$

Example. What is $\dfrac{d}{dx}(c)$? What is $\dfrac{d}{dx}(x)$?

Theorem (Power Rule)**.** If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example. If $f(x) = x^6$, what is $f'(x)$?

Example. If $y = x^{1000}$, what is y'?

Example. If $y = t^4$, what is $\frac{dy}{dt}$?

Example. What is $\frac{d}{dr}(r^3)$?

Example. What is $\frac{d}{dx}\left(\frac{1}{x}\right)$?

Example. What is $\frac{d}{dx}(\sqrt{x})$?

Example. Differentiate $f(x) = \frac{1}{x^2}$.

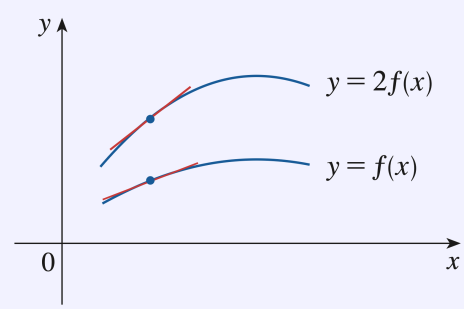
Example. Differentiate $y = \sqrt[3]{x^2}$.

Example. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Theorem (Constant Multiple Rule)**.** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\big[cf(x)\big] \;=\; c\,\frac{d}{dx}f(x).$$

Geometric Interpretation: Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are also doubled.



Example.

- What is $\dfrac{d}{dx}\left(3x^4\right)$?

- What is $\dfrac{d}{dx}\left(-x\right)$?

Theorem (Sum and Difference Rules)**.** If f and g are both differentiable, then their derivatives satisfy:

Leibniz Notation	**Prime Notation**
$\dfrac{d}{dx}\big[f(x)+g(x)\big] \;=\; \dfrac{d}{dx}f(x)+\dfrac{d}{dx}g(x)$	$(f+g)'(x) \;=\; f'(x)+g'(x)$
$\dfrac{d}{dx}\big[f(x)-g(x)\big] \;=\; \dfrac{d}{dx}f(x)-\dfrac{d}{dx}g(x)$	$(f-g)'(x) \;=\; f'(x)-g'(x)$

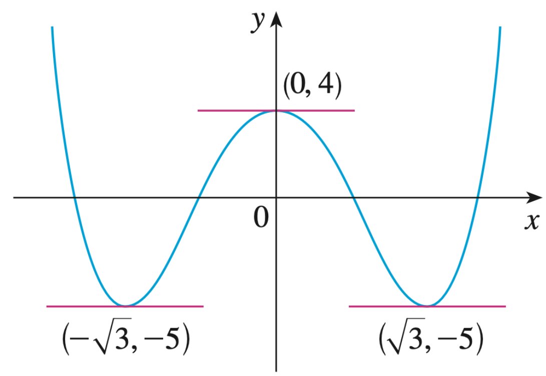
Proof. To prove the Sum Rule, we let $F(x) = f(x) + g(x)$. Then

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\[2em]
&= \lim_{h\to 0} \frac{\qquad\qquad\qquad\qquad\qquad\qquad}{h} \\[2em]
&= \lim_{h\to 0} \left[\frac{\qquad\qquad\qquad}{h} + \frac{\qquad\qquad\qquad}{h}\right] \\[2em]
&= \lim_{h\to 0} \frac{\qquad\qquad\qquad}{h} + \lim_{h\to 0} \frac{\qquad\qquad\qquad}{h} \\[2em]
&= f'(x) + g'(x).
\end{aligned}
$$

$\square$

Example. What is $\dfrac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$?

Example. Where does the curve $y = x^4 - 6x^2 + 4$ have horizontal tangent lines?



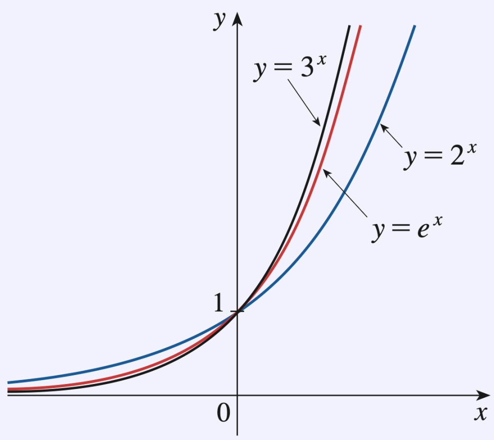
Example. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

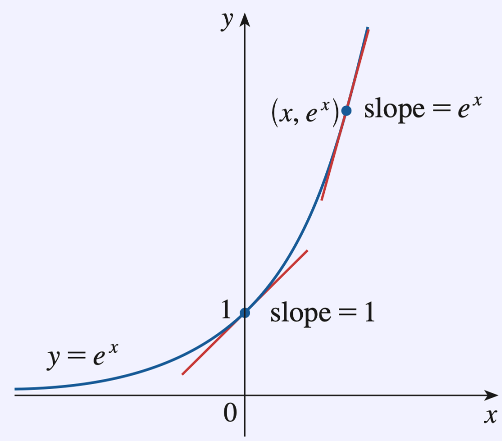
Definition. The number e is the unique base of the exponential function for which the derivative at $x = 0$ is 1:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

It follows that e is the base for which the exponential function is its own derivative:

$$\frac{d}{dx}\left(e^x\right) = e^x.$$





Numerically, $e \approx 2.71828$.

Consider the general exponential function $f(x) = b^x$. Using the definition of the derivative,

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\qquad\qquad}{h} &= \lim_{h\to 0} \frac{\qquad\qquad}{h} \\ &= \lim_{h\to 0} \frac{\qquad\qquad}{h} \\ &= \qquad \cdot \lim_{h\to 0} \frac{b^h - 1}{h} \end{aligned}$$

Thus $f'(x) = b^x \cdot f'(0)$. Numerical evidence shows:

$$\begin{aligned} f'(0) &\approx \qquad\qquad && \text{for } b = 2, \\ f'(0) &\approx \qquad\qquad && \text{for } b = 3. \end{aligned}$$

There must be some base b between 2 and 3 for which $f'(0) = 1$. We denote this special base by e.

Example. If $f(x) = e^x - x$, find f' and f''.

Example. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?