3.1 Derivatives of Polynomials and Exponential Functions

Definition. With respect to the variable x, the symbol

$$\frac{d}{dx}$$

means "*take the derivative with respect to* x." It is an *operator* that takes a function f as input and produces the derivative function as output. The functions

$$\underbrace{\frac{d}{dx}\big[f(x)\big]}_{\text{operator on } f} \;=\; \underbrace{f'(x)}_{\text{Lagrange}} \;=\; \underbrace{\frac{df}{dx}}_{\text{Leibniz}}$$

all denote the same derivative function. If $y = f(x)$, then

$$\frac{dy}{dx} \;=\; f'(x) \;=\; \frac{d}{dx}\big[f(x)\big].$$

In contexts where the independent variable is *time* t, one often uses Newton's dot notation:

$$\dot{y} \;=\; \frac{dy}{dt}, \qquad \ddot{y} \;=\; \frac{d^2y}{dt^2}.$$

The second and nth derivatives can be written as

$$f''(x) \;=\; \frac{d^2f}{dx^2} \;=\; \frac{d^2}{dx^2}\big[f(x)\big] \qquad f^{(n)}(x) \;=\; \frac{d^nf}{dx^n} \;=\; \frac{d^n}{dx^n}\big[f(x)\big]$$

Example. What is $\dfrac{d}{dx}(c)$? What is $\dfrac{d}{dx}(x)$?

Constant functions have slope 0: $\dfrac{d}{dx}(c) = 0$

$f(x) = x$ has slope 1: $\dfrac{d}{dx}(x) = 1$

Theorem (Power Rule)**.** If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example. If $f(x) = x^6$, what is $f'(x)$?

$\boxed{6x^5}$

Example. If $y = x^{1000}$, what is y'?

$\boxed{1000x^{999}}$

Example. If $y = t^4$, what is $\frac{dy}{dt}$?

$\boxed{4t^3}$

Example. What is $\frac{d}{dr}(r^3)$?

$\boxed{3r^2}$

Example. What is $\frac{d}{dx}\left(\frac{1}{x}\right)$?

$\frac{1}{x} = x^{-1}$

$-1 \cdot x^{-2} = \boxed{\frac{-1}{x^2}}$

Example. What is $\frac{d}{dx}(\sqrt{x})$?

$\sqrt{x} = x^{1/2}$

$\frac{1}{2} \cdot x^{-1/2} = \boxed{\frac{1}{2\sqrt{x}}}$

Example. Differentiate $f(x) = \frac{1}{x^2}$.

$\frac{1}{x^2} = x^{-2}$

$-2 \cdot x^{-3} = \boxed{\frac{-2}{x^3}}$

Example. Differentiate $y = \sqrt[3]{x^2}$.

$\sqrt[3]{x^2} = x^{2/3}$

$\frac{2}{3} x^{-1/3} = \boxed{\frac{2}{3\sqrt[3]{x}}}$

Example. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Slope at $(1,1)$:

Since $y = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$

$\boxed{y' = \frac{3}{2} x^{1/2}}$ (by the power rule)

When $x=1$, $y' = \frac{3}{2}(1)^{1/2} = \frac{3}{2}$

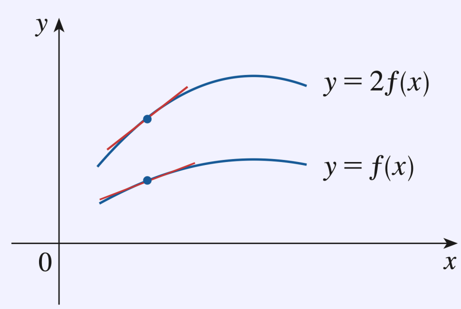
Using point-slope, the equation of the tangent line is

$\boxed{y-1 = \frac{3}{2}(x-1)}$

Theorem (Constant Multiple Rule)**.** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\left[cf(x)\right] \;=\; c\,\frac{d}{dx}f(x).$$

Geometric Interpretation: Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are also doubled.



Example.

- What is $\frac{d}{dx}(3x^4)$? $= 3 \cdot \frac{d}{dx}(x^4) = 3 \cdot (4x^3) = \boxed{12x^3}$
- What is $\frac{d}{dx}(-x)$? $= -1 \cdot \frac{d}{dx}(x) = -1 \cdot (1) = \boxed{-1}$

Theorem (Sum and Difference Rules)**.** If f and g are both differentiable, then their derivatives satisfy:

Leibniz Notation	**Prime Notation**
$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$	$(f+g)'(x) = f'(x) + g'(x)$
$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$	$(f-g)'(x) = f'(x) - g'(x)$

Proof. To prove the Sum Rule, we let $F(x) = f(x) + g(x)$. Then

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right]$$

the sum property for limits

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$

$\square$

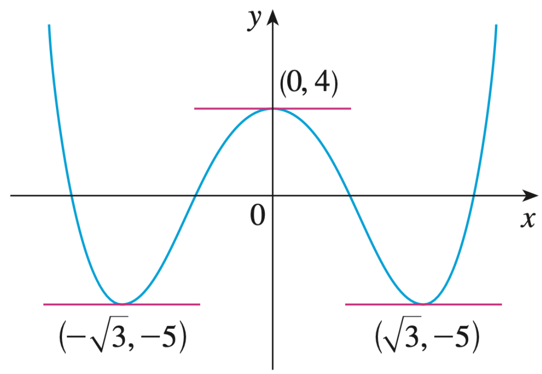
Example. What is $\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$?

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Example. Where does the curve $y = x^4 - 6x^2 + 4$ have horizontal tangent lines?

Solve for where the derivative is 0



$$y' = 4x^3 - 12x$$

$$\Rightarrow 4x^3 - 12x = 0$$

$$\Rightarrow 4x(x^2 - 3) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \pm\sqrt{3}$$

Plug these values back into $y = x^4 - 6x^2 + 4$ to get the y-coords.

$$\boxed{(-\sqrt{3}, -5), \quad (0, 4), \quad (\sqrt{3}, -5)}$$

(distance function)

Example. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

$$s(t) = 2t^3 - 5t^2 + 3t + 4$$

$$\frac{ds}{dt} = v(t) = 6t^2 - 10t + 3$$

$$\frac{d^2s}{dt^2} = a(t) = 12t - 10$$

When $t = 2$, $\boxed{a(2) = 24 - 10 = 14 \text{ cm}/s^2}$

(velocity is cm/s, acceleration is $\frac{\text{cm/s}}{\text{s}}$)

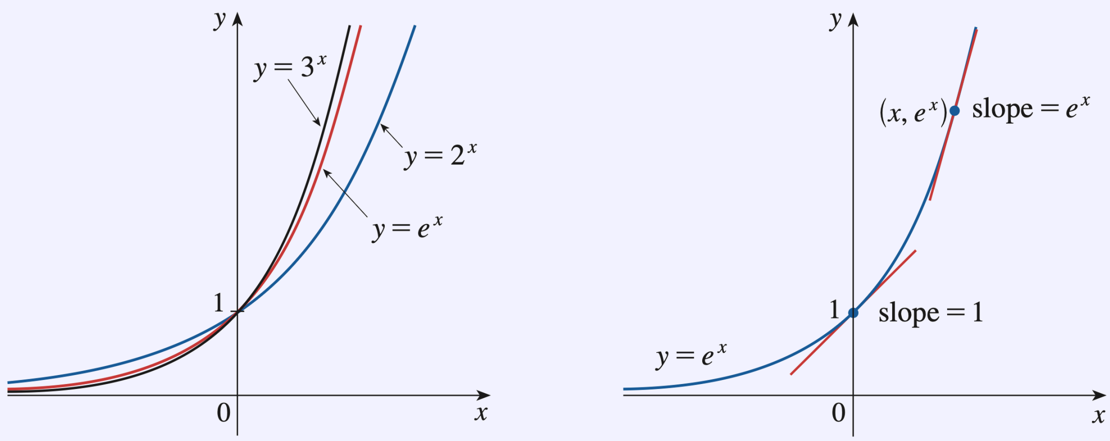
Definition. The number e is the unique base of the exponential function for which the derivative at $x = 0$ is 1:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

The derivative of e^x at 0 is 1

It follows that e is the base for which the exponential function is its own derivative:

$$\frac{d}{dx}\left(e^x\right) = e^x.$$



Numerically, $e \approx 2.71828$.

Consider the general exponential function $f(x) = b^x$. Using the definition of the derivative,

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} \\ &= b^x \lim_{h \to 0} \frac{b^h - 1}{h} \end{aligned}$$

This is $f'(0)$

Thus $f'(x) = b^x \cdot f'(0)$. Numerical evidence shows:

$$f'(0) \approx 0.693 \qquad \text{for } b = 2,$$

$$f'(0) \approx 1.099 \qquad \text{for } b = 3.$$

There must be some base b between 2 and 3 for which $f'(0) = 1$. We denote this special base by e.

Example. If $f(x) = e^x - x$, find f' and f''.

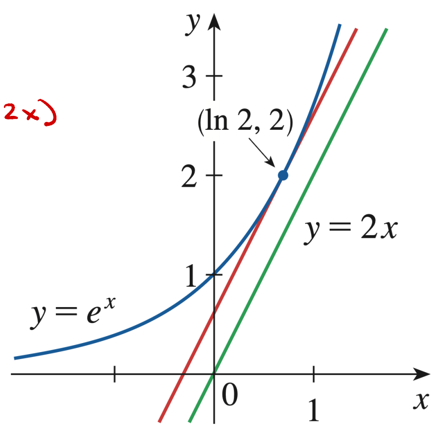
$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = \boxed{e^x - 1}$$

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = \boxed{e^x}$$

Example. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Goal: Find x where $y' = 2$

(so the tangent line has the same slope as $y = 2x$)



$y' = e^x \quad \Rightarrow \quad e^x = 2$

$\Rightarrow \quad x = \ln 2$

Point: $(\ln 2, e^{\ln 2}) = \boxed{(\ln 2, 2)}$