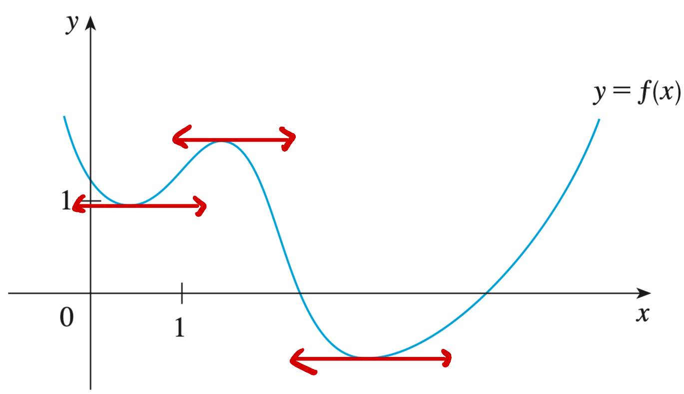
2.8 The Derivative as a Function

Definition. The derivative of a function f at a number x is defined by

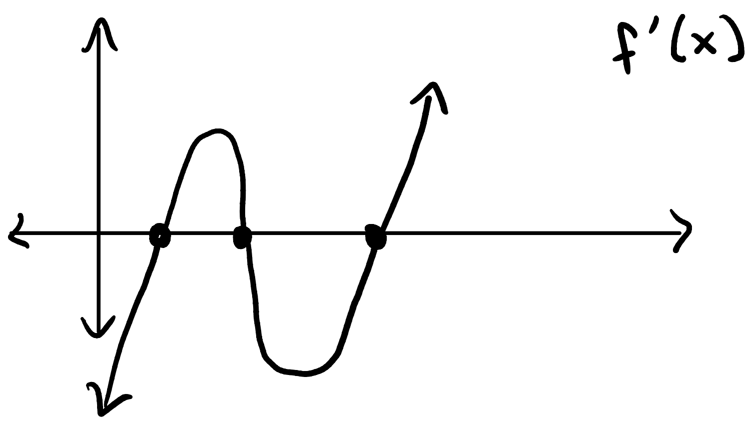
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. The value of f' at x, namely $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example. Below is the graph of a function $f(x)$. Sketch the graph of the derivative $f'(x)$.

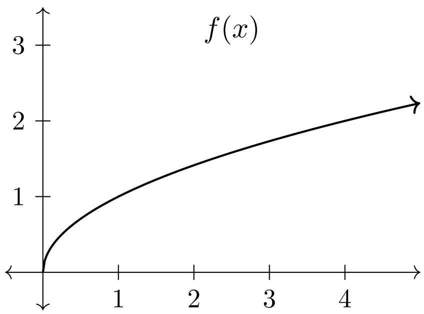


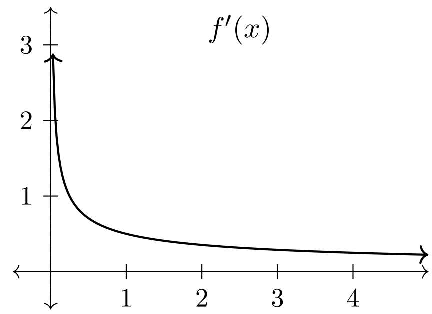
Strategy: ① Look at where the derivative is 0.
② Decide if the derivative is pos/neg between



Quiz!

Example. If $f(x) = \sqrt{x}$, find the derivative of f. What is the domain of f'?





$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

this is 1

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

The domain of $f(x)$ is $[0, \infty)$

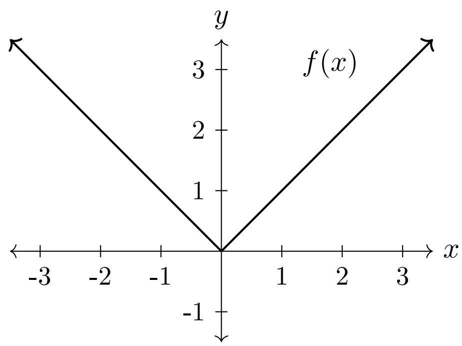
The domain of $f'(x)$ is $(0, \infty)$

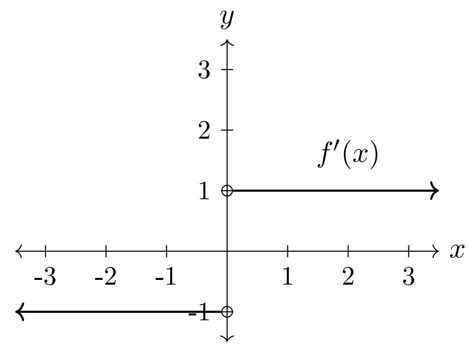
As you go to 0, you approach a vertical tangent.

$\left(\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \infty, \right.$ so f' not defined$)$

Definition. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) $\big[$or (a, ∞), or $(-\infty, a)$, or $(-\infty, \infty)\big]$ if it is differentiable at every number in the interval.

Example. Where is the function $f(x) = |x|$ differentiable?





If $x > 0$, the slope is 1

If $x < 0$, the slope is -1

At $x = 0$, $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

To show this doesn't exist, we will look at the one-sided limits.

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = \boxed{1}$$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = \boxed{-1}$$

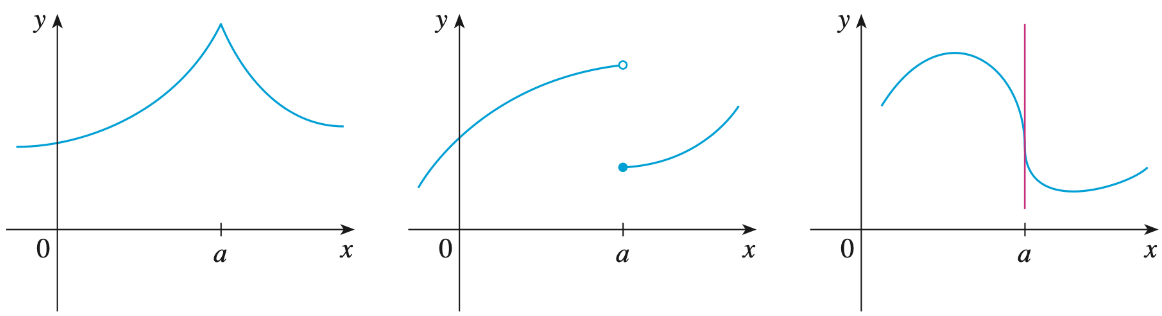
These are different, so $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ DNE $\Rightarrow$ $\boxed{f(x) \text{ is not differentiable at } 0}$

Theorem. What can we say if f is differentiable at a?

If $f(x)$ is differentiable at a point a,
you can show it has to be continuous at a.

[Not obvious. Needs a proof.]

Question. How can a function fail to be differentiable?



(a) A corner | (b) A discontinuity | (c) A vertical tangent

The left/right handed limits will be different so the limit will fail to exist.
(like $|x|$ example)

if you are differentiable, you are continuous

the derivative goes off to infinity and you'll get an asymptote
(like the $\sqrt{x}$ example)