2.7 Derivatives and Rates of Change

Question. What is the story so far?

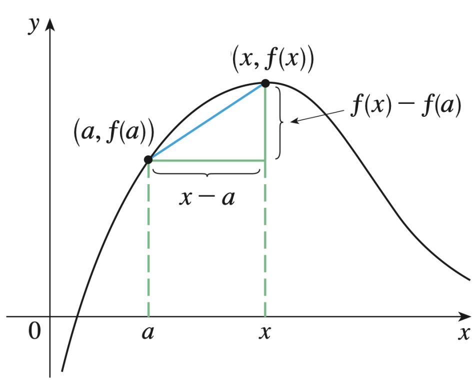
§2.1 : We want to find instantaneous velocity

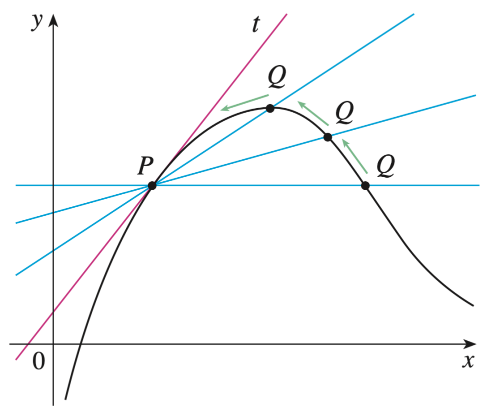
§2.2 - 2.6 : We needed to make the "limiting procedure" more precise

§2.7 : We can actually find instanteneous velocity

Definition.

- What is the slope of the secant line through $(a, f(a))$ and $(x, f(x))$?
- What is the slope of the tangent line at $(a, f(a))$?





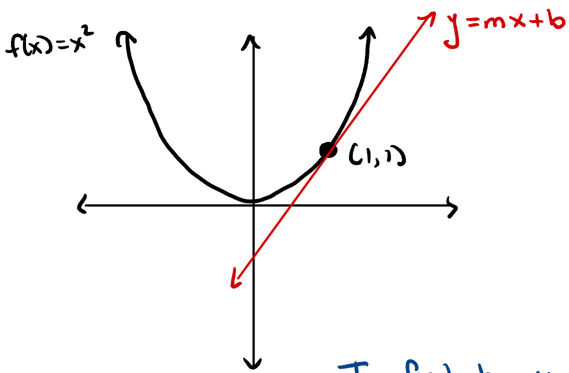
① $\dfrac{\Delta y}{\Delta x} = \boxed{\dfrac{f(x) - f(a)}{x - a}}$

- The average rate of change on $[a, x]$
- The slope of the secant line

② $\boxed{\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}}$

- Move x extremely close to a to get instantaneous change
- The slope of the tangent line

Example. Find an equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.



To find the slope:

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1}$$

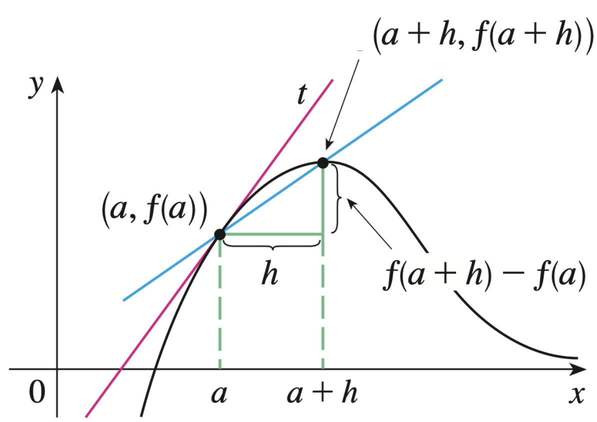
$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1} x+1 = 2$$

To find b, use the point (1,1):

In point-slope form $\boxed{y-1 = 2(x-1)}$ ✓ $y-y_1 = m(x-x_1)$

Alternatively, $\boxed{y = 2x-1}$

Definition. What is another way to write the slope of the tangent line through $(a, f(a))$? ALTERNATE PERSPECTIVE



Here, h represents the length of the interval on which you find the slopes of secant lines

① $\dfrac{\Delta y}{\Delta x} = \boxed{\dfrac{f(a+h)-f(a)}{h}}$

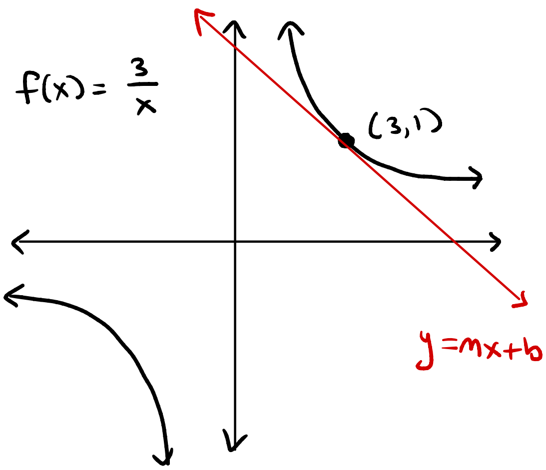
- Average rate of change on $[a, a+h]$
- The slope of the secant line

② $\boxed{\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}}$

- Move h closer and closer to 0 to get instantaneous change
- The slope of the tangent line

Typically easier to work with.

Example. Find an equation of the tangent line to the hyperbola $y = \dfrac{3}{x}$ at the point $(3, 1)$.



To find the slope:

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\left[\frac{3}{3+h}\right] - [1]}{h}$$

$$= \lim_{h\to 0} \frac{\left[\frac{3}{3+h}\right] - \left[\frac{3+h}{3+h}\right]}{h} = \lim_{h\to 0} \frac{\left[\frac{-h}{3+h}\right]}{\left[\frac{h}{1}\right]}$$

$$= \lim_{h\to 0} \frac{-h}{3+h} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{3+h} = \frac{-1}{3}$$

Equation: $\boxed{y - 1 = \frac{-1}{3}(x-3)}$

Example. Suppose that a ball is dropped from 450 m above the ground.

- What is the velocity of the ball after 5 seconds?
- How fast is the ball traveling when it hits the ground?

Recall: $f(t) = 4.9t^2$ is how far the ball has fallen at time t (§2.1 Notes)

Let's find the slope of the tangent line at any point $t=a$.

↳ instantaneous change

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[4.9(a+h)^2]-[4.9a^2]}{h}$$

$$= \lim_{h\to 0} \frac{4.9(a^2+2ah+h^2)-4.9a^2}{h} = \lim_{h\to 0} \frac{9.8a\cancel{h}+4.9h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h\to 0} 9.8a + 4.9h = 9.8a$$

* This shows that the instantaneous velocity at time $t=a$ is $9.8a$

When $t=5$, the velocity is $9.8(5) = \boxed{49 \text{ m/s}}$

For the 2nd Question, when does it hit the ground?

$$f(t) = 4.9t^2 = 450 \Rightarrow t = 9.6 \text{ seconds}$$

Hence, $9.8(9.6) = \boxed{94.08 \text{ m/s}}$ when it hits the ground.

Definition. What is the derivative of a function f at a number a?

Define a new function $f'(a)$

$\boxed{\text{At any point } a} \rightarrow \boxed{f'} \Longrightarrow$ The slope of the tangent line to $f(x)$ at the point a

The derivative function

Example. Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{[x^2 - 8x + 9] - [a^2 - 8a + 9]}{x - a}$$

$$\lim_{x \to a} \frac{x^2 - a^2 - 8x + 8a}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} - \frac{8x - 8a}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}} - \frac{8\cancel{(x-a)}}{\cancel{(x-a)}} = \lim_{x \to a} x + a - 8 = 2a - 8$$

$$\boxed{f'(a) = 2a - 8}$$

Example. Find an equation of the tangent line to the parabola $f(x) = x^2 - 8x + 9$ at the point $(3, -6)$.