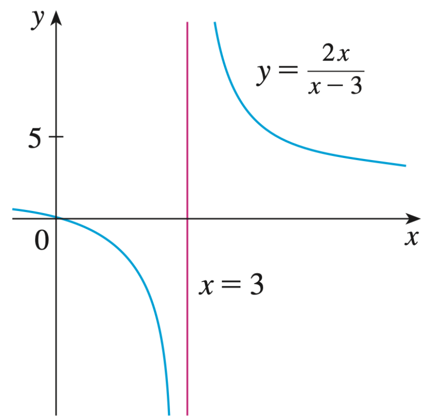
2.6 Limits at Infinity & Horizontal Asymptotes

Definition. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example. Find $\displaystyle\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\displaystyle\lim_{x\to 3^-} \frac{2x}{x-3}$

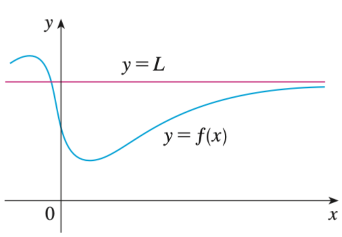


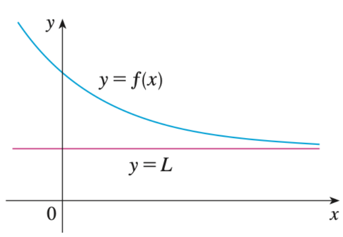
Definition. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

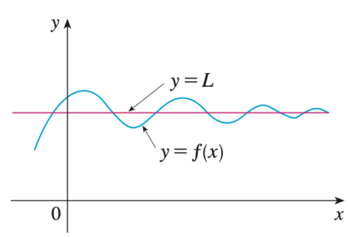
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

This means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large in either the positive or negative direction.

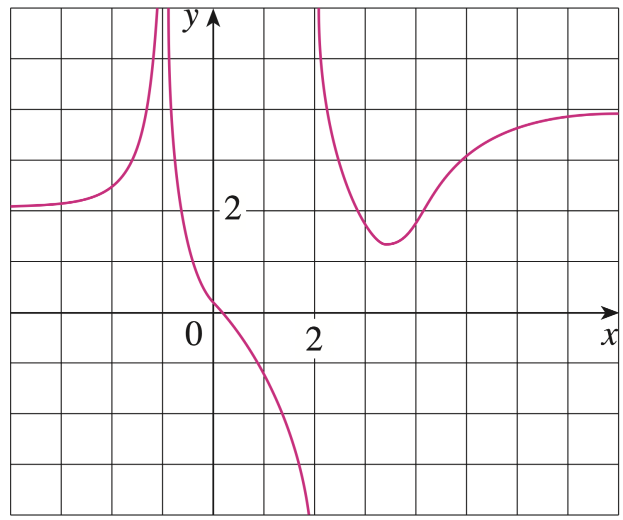
Example. Here are some examples of horizontal asymptotes







Example. Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown below



Example. Find $\lim\limits_{x\to\infty} \dfrac{1}{x}$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x}$

Remark. What can we say about $\dfrac{1}{x^n}$?

Example. Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Example. Compute $\lim_{x\to\infty} (\sqrt{x^2+1} - x)$

Example. Show that $\lim_{x\to-\infty} e^x = 0$. Then find $\lim_{x\to 0^-} e^{1/x}$.

Example. Evaluate $\lim_{x\to\infty} \sin x$.

Example. Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.