2.6 Limits at Infinity & Horizontal Asymptotes

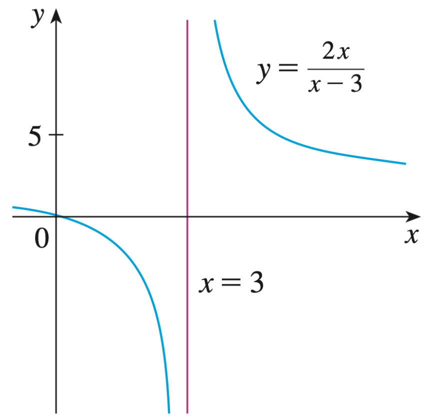
Definition. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example. Find $\displaystyle\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\displaystyle\lim_{x\to 3^-} \frac{2x}{x-3}$

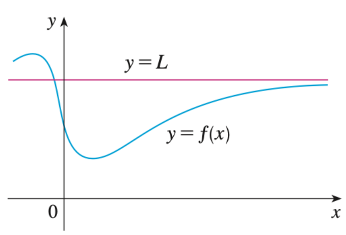
$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$$

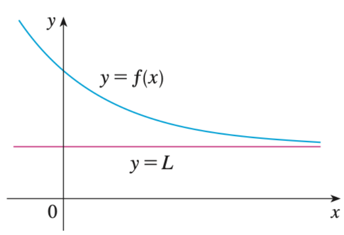
$$\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$$

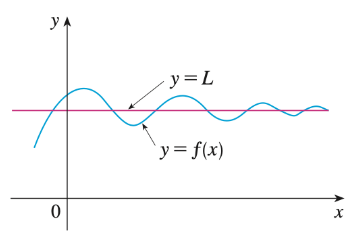


Definition. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$
This means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large in either the positive or negative direction.

Example. Here are some examples of horizontal asymptotes

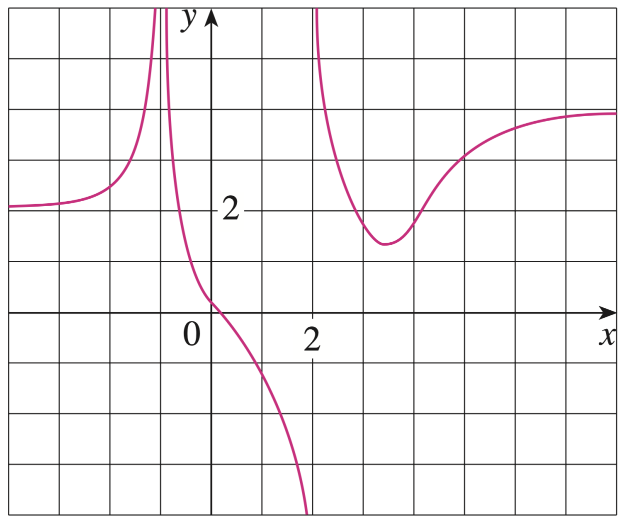






In all three cases, $\displaystyle\lim_{x\to\infty} f(x) = L$

Example. Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown below



Vertical Asymptotes:

$x = -1$ because $\lim_{x \to -1} f(x) = \infty$

$x = 2$ because $\lim_{x \to 2^+} f(x) = \infty$

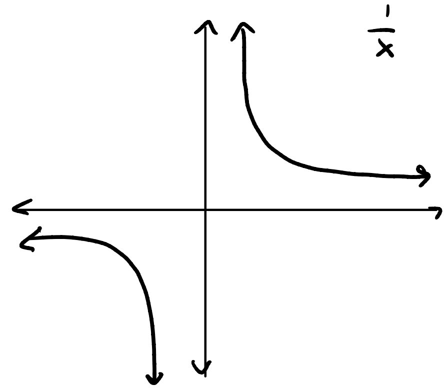
$\lim_{x \to 2^-} f(x) = -\infty$

Horizontal Asymptotes:

$y = 2$ because $\lim_{x \to -\infty} f(x) = 2$

$y = 4$ because $\lim_{x \to \infty} f(x) = 4$

Example. Find $\lim\limits_{x\to\infty} \dfrac{1}{x}$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x}$



$\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

$\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

Remark. What can we say about $\dfrac{1}{x^n}$?

$\lim\limits_{x \to \infty} \dfrac{1}{x^n} = 0$

$\lim\limits_{x \to -\infty} \dfrac{1}{x^n} = 0$

Example. Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1} \cdot \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

This

all of these terms go to 0 as $x \to \infty$

Using the sum/quotient laws for limits

$$= \frac{3-0-0}{5+0+0} = \boxed{\frac{3}{5}}$$

(I chose $1/x^2$ because 2 is the highest power appearing in the denominator)

Example. Compute $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x)$

WRONG: $\lim\limits_{x\to\infty} \sqrt{x^2+1} - \lim\limits_{x\to\infty} x = \infty - \infty = 0$ ✗

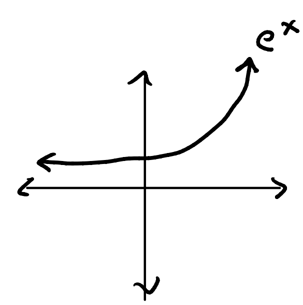
(To use the difference law, both limits need to exist)

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \cdot \left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)$$

This is 1

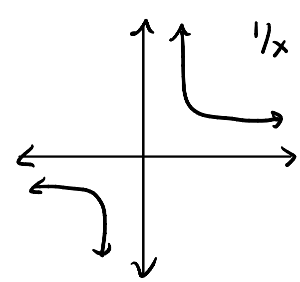
$$= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = \boxed{0}$$

Example. Show that $\lim\limits_{x\to-\infty} e^x = 0$. Then find $\lim\limits_{x\to 0^-} e^{1/x}$.



① Looking at the graph of e^x,

$$\lim_{x\to-\infty} e^x = 0$$

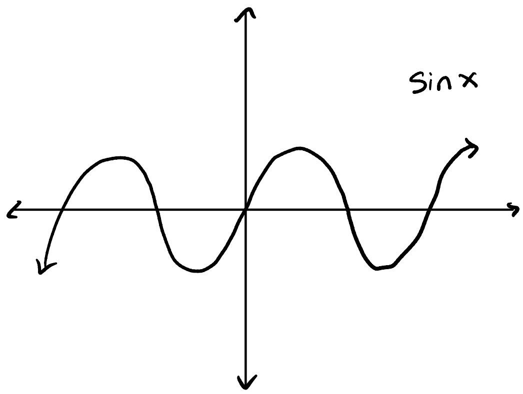


② Substitute $t = \frac{1}{x}$

As $x \to 0^-$, $t \to -\infty$

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0$$

Example. Evaluate $\lim_{x\to\infty} \sin x$.



The graph oscillates back and forth forever and doesn't approach anything

$$\boxed{\lim_{x\to\infty} \sin x = \text{D.N.E.}}$$

Example. Find $\lim_{x\to\infty} \frac{x^2+x}{3-x}$.

Evaluating $\lim_{x\to\pm\infty}$ of Rational Functions:

1. If the largest power is on top $\to \pm\infty$
2. If powers are the same $\to$ ratio of coefficients
3. If the largest power is on bottom $\to 0$

This is 1

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} \cdot \left(\frac{1/x}{1/x}\right)$$

$$= \lim_{x\to\infty} \frac{x+1}{\underbrace{\frac{3}{x}}_{\text{goes to } 0} - 1}$$

$\therefore$ the top goes to ∞

the bottom goes to -1

$$\boxed{-\infty}$$