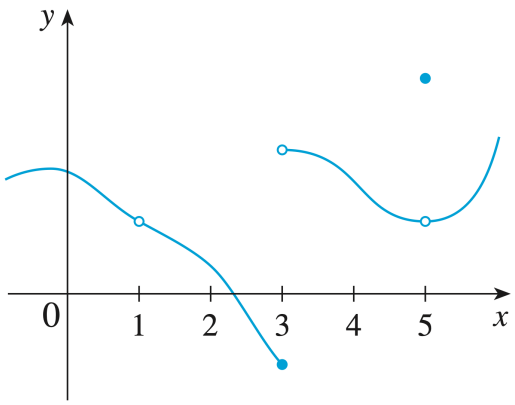
2.5 Continuity

Definition. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Example. Where is f discontinuous?



Example. Let

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

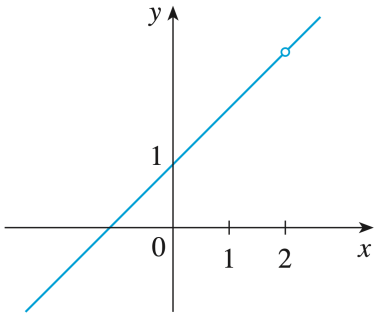
Why is $f(x)$ not continuous at 0?

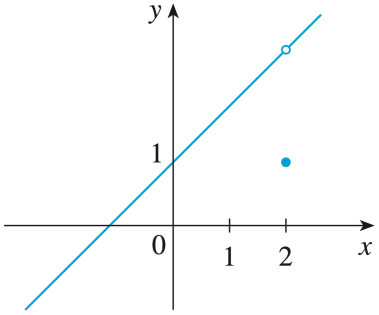
Example. Let

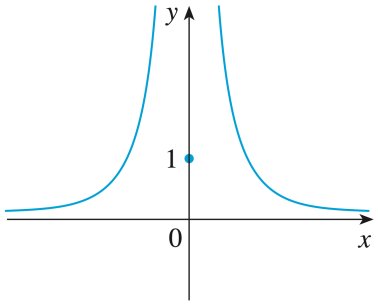
$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

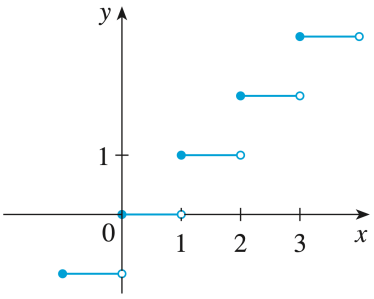
Why is $f(x)$ not continuous at 2?

Definition. What are the different types of discontinuities?









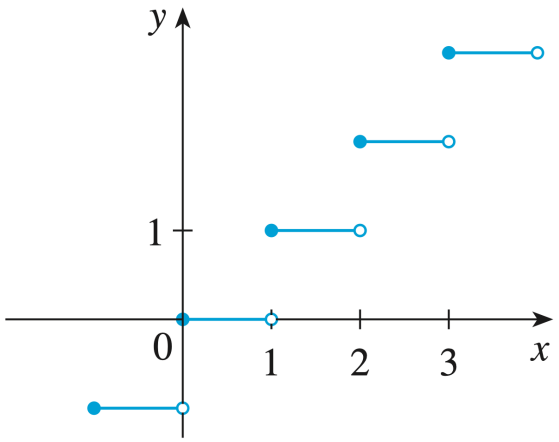
Definition. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a).$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example. At each integer n, is this function continuous from the left? Is it continuous from the right?



Definition. A function f is continuous on an interval if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.

Example. Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Remark. Instead of always using the definition to verify the continuity of a function, it's convenient to build up complicated continuous functions from simple ones.

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

- $f + g$
- $f - g$
- cf
- fg
- $\dfrac{f}{g}$ if $g(a) \neq 0$

Proof.

□

Question. Why is any polynomial continuous on $(-\infty, \infty)$?

Question. Why is any rational function continous on its domain?

Example. Find $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Theorem. The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Example. Where is the function $f(x) = \dfrac{\ln x + e^x}{x^2 - 1}$ continuous?

Example. What is $\lim\limits_{x \to \pi} \dfrac{\sin x}{2 + \cos x}$?

Theorem. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

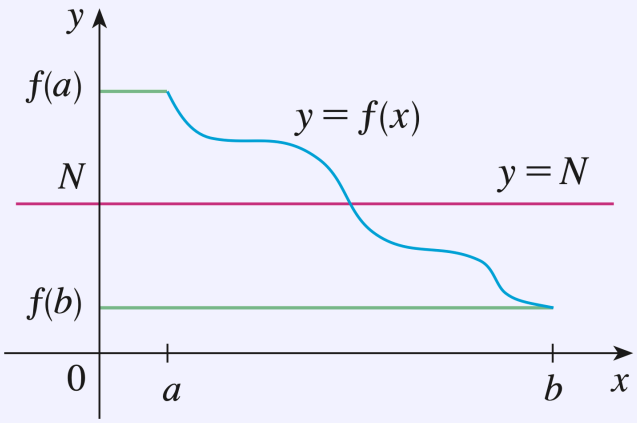
$$\lim_{x\to a} f(g(x)) \;=\; f\Big(\lim_{x\to a} g(x)\Big) = f(b).$$

Corollary. If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Example. Where is $h(x) = \sin(x^2)$ continuous?

Example. Where is $F(x) = \ln(1 + \cos x)$ continuous?

Theorem (Intermediate Value Theorem)**.** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.



Remark. Why did the chicken cross the road?

Example. Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.