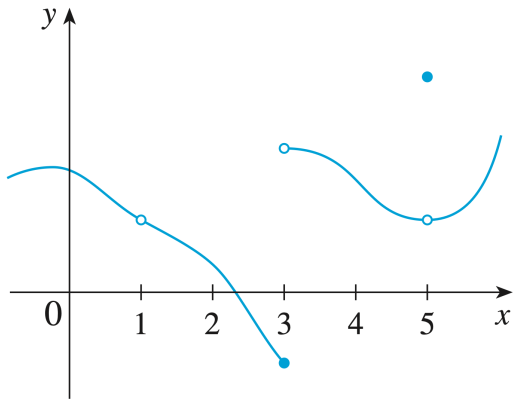
2.5 Continuity

This means you can draw the graph of f(x) without picking up your pencil at x = a.

Definition. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Example. Where is f discontinuous?



At $x=1$: the function is undefined

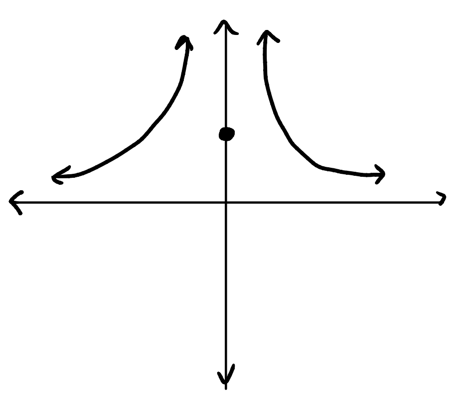
At $x=3$: the limit is D.N.E.

At $x=5$: the value of the function is different than the limit (but both exist)

Example. Let

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Why is $f(x)$ not continuous at 0?



Compare $f(0)$ and $\lim_{x \to 0} f(x)$

$f(0) = 1$ and $\lim_{x \to 0} f(x) = \infty$

$\Rightarrow$ Not continuous since $f(0) \neq \lim_{x \to 0} f(x)$

Example. Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

Why is $f(x)$ not continuous at 2?

Compare $\lim_{x \to 2} f(x)$ and $f(2)$

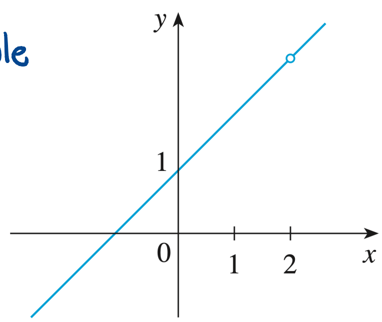
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = \lim_{x \to 2} x + 1 = 3$$

But $f(2) = 1$.

Not continuous, since $\lim_{x \to 2} f(x) \neq f(2)$

Definition. What are the different types of discontinuities?

Removable



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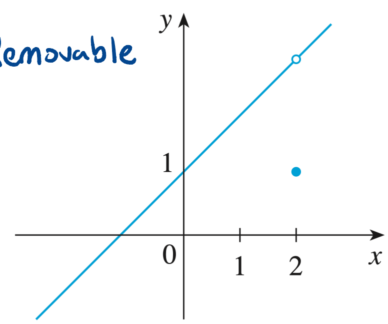
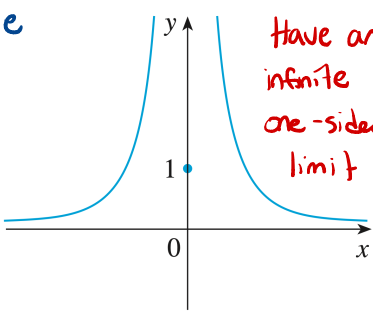


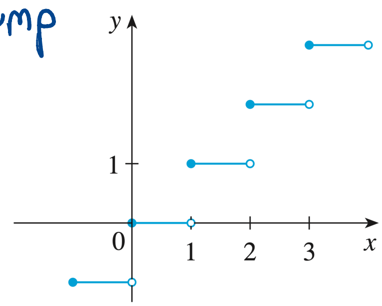
$\lim_{x \to a} f(x)$ exists

Infinite



Have an infinite one-sided limit

Jump



$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

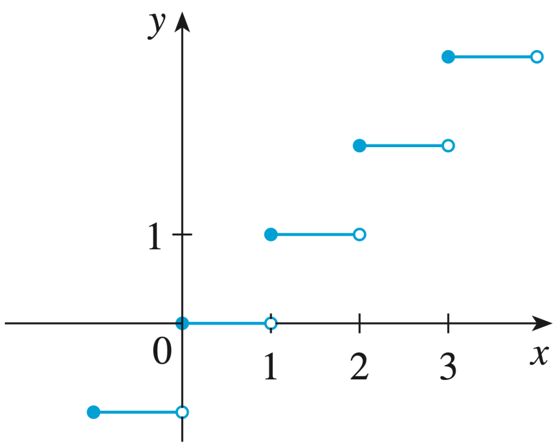
Definition. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a).$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example. At each integer n, is this function continuous from the left? Is it continuous from the right?



$f(n) = n$

$\lim_{x \to n^+} f(x) = n$

$\lim_{x \to n^-} f(x) = n-1$

$\Rightarrow$ This function is right-continuous but not left-continuous

Definition. A function f is continuous on an interval if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.

Example. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Let $-1 < a < 1$

$$\lim_{x\to a} f(x) = \lim_{x\to a} 1 - \sqrt{1-x^2} \overset{\text{Difference Law}}{=} \lim_{x\to a} 1 - \lim_{x\to a}\sqrt{1-x^2} \overset{\text{Root Law}}{=} 1 - \sqrt{\lim_{x\to a} 1-x^2}$$

Difference Law $\rightarrow$ $$= 1 - \sqrt{\lim_{x\to a} 1 - \lim_{x\to a} x^2}$$

$$= 1 - \sqrt{1-a^2}$$

$$= f(a)$$

(can do endpoints with one-sided limits)

Remark. Instead of always using the definition to verify the continuity of a function, it's convenient to build up complicated continuous functions from simple ones.

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

- $f + g$
- $f - g$
- cf
- fg
- $\dfrac{f}{g}$ if $g(a) \neq 0$

Proof.

Let's prove #1: show that $f+g$ is continuous at a

Since $f(x)$ is continuous at a, $\lim_{x\to a} f(x) = f(a)$

Since $g(x)$ is continuous at a, $\lim_{x\to a} g(x) = g(a)$

Sum property for limits

Hence $$\lim_{x\to a}\left[(f+g)(x)\right] = \lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$= f(a) + g(a)$$

$$= (f+g)(a)$$ $\square$

This equality shows $f+g$ is continuous

e.g. $5x^6 + 4x^5 + 2x^2 + x + 5$

Question. Why is any polynomial continuous on $(-\infty, \infty)$?

Polynomials are of the form $f(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$

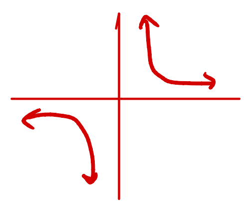
These are continuous everywhere because each individual $c_i x^i$ is continuous everywhere, and sums of continuous functions are continuous.

Question. Why is any rational function ~~continous~~ continuous on its domain?

Rational functions are of the form $\dfrac{P(x)}{Q(x)}$ where P and Q are polynomials.

The domain is $Q(x) \neq 0$. Quotients of continuous functions are continuous where the denominator is nonzero ✓

e.g. $\dfrac{1}{x}$



Not continuous at 0, but continuous on its domain

Example. Find $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

This is a rational function with domain $x \neq \frac{5}{3}$

So it is continuous at $x = -2$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \boxed{\frac{-1}{11}}$$

we did this in §2.3

Theorem. The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Example. Where is the function $f(x) = \dfrac{\ln x + e^x}{x^2 - 1}$ continuous?

1. $\ln x$ is continuous on its domain $x > 0$
2. e^x is continuous on $(-\infty, \infty)$
3. $x^2 - 1$ is continuous on $(-\infty, \infty)$
4. $\ln x + e^x$ is continuous on $(0, \infty)$
5. $\dfrac{\ln x + e^x}{x^2 - 1}$ is continuous on $(0, \infty)$ whenever $x^2 - 1 \neq 0$ (i.e. $x \neq \pm 1$)

$$\boxed{\text{Final Answer: } (0,1) \cup (1, \infty)}$$

Example. What is $\lim\limits_{x \to \pi} \dfrac{\sin x}{2 + \cos x}$?

$\sin x$ is continuous on $(-\infty, \infty)$

$2 + \cos x$ is continuous on $(-\infty, \infty)$

$\Rightarrow$ $\dfrac{\sin x}{2 + \cos x}$ is continuous on $(-\infty, \infty)$ as long as $2 + \cos x \neq 0$ (it never is 0)

$$\Rightarrow \lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin(\pi)}{2 + \cos(\pi)} = \frac{0}{2 + (-1)} = 0$$

Evaluating limits of compositions

Theorem. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) \;=\; f\Big(\lim_{x\to a} g(x)\Big) = f(b).$$

Corollary. If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Where is a composition continuous?

Example. Where is $h(x) = \sin(x^2)$ continuous?

$g(x) = x^2$ is continuous on $(-\infty, \infty)$

$f(x) = \sin(x)$ is continuous on $(-\infty, \infty)$

$h(x) = f(g(x)) = \sin(x^2)$ is also continuous on $(-\infty, \infty)$

Example. Where is $F(x) = \ln(1 + \cos x)$ continuous?

$g(x) = 1 + \cos x$ is continuous on $(-\infty, \infty)$

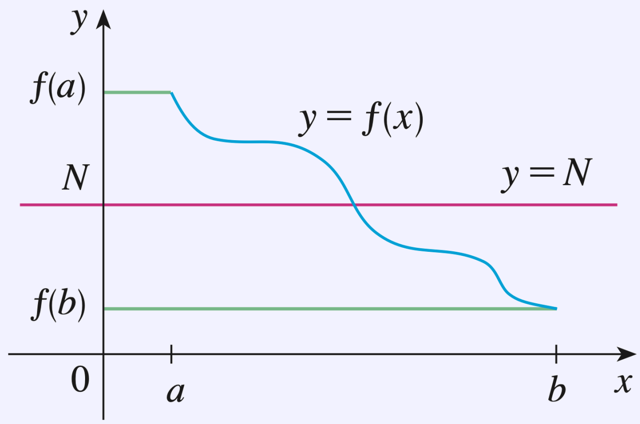
But $f(x) = \ln x$ is only continuous for $x > 0$

$\Rightarrow$ $F(x) = f(g(x))$ is continuous for $1 + \cos x > 0$

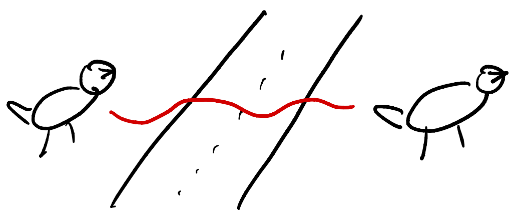
$1 + \cos x > 0$ except at $\cdots -3\pi, -\pi, \pi, 3\pi, \cdots$ odd multiples of π

$$\boxed{\cdots \cup (-3\pi, -\pi) \cup (-\pi, \pi) \cup (\pi, 3\pi) \cup \cdots}$$

Theorem (Intermediate Value Theorem)**.** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.



Remark. Why did the chicken cross the road?



If the chicken were on both sides of the road, it had to have crossed it by I.V.T

Example. Show that there is a root of the equation

$$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

$f(1) = -1$ and $f(2) = 12$

* Must explain continuity on an exam to apply IVT

$f(x)$ is a polynomial, which is continuous on $[1, 2]$.

Since $f(1) < 0$ and $f(2) > 0$, there is some value c in $[1,2]$ where $f(c) = 0$ by I.V.T.