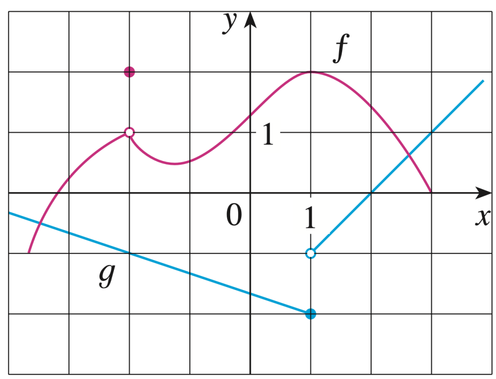
2.3 Calculating Limits Using the Limit Laws

Theorem. Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

1. $\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. Sum Law
2. $\lim_{x\to a}\big(f(x)-g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. Difference Law
3. $\lim_{x\to a}\big(c\,f(x)\big) = c\lim_{x\to a} f(x)$. Constant Multiple Law
4. $\lim_{x\to a}\big(f(x)\,g(x)\big) = \Big(\lim_{x\to a} f(x)\Big)\cdot\Big(\lim_{x\to a} g(x)\Big)$. Product Law
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$. Quotient Law

Remark. All of these hold for one-sided limits as well.

Example. Use the Limit Laws and the graphs of f and g to evaluate the following limits.



- $\lim_{x\to -2}[f(x)+5g(x)]$

- $\lim_{x\to 1}[f(x)g(x)]$

- $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$

Theorem. Suppose the limit $\lim_{x \to a} f(x)$ exists. Then:

1. $\lim_{x \to a} \Big[f(x) \Big]^n = \Big[\lim_{x \to a} f(x) \Big]^n$ — Power Law
2. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ — Root Law

Example. Evaluate the following limits and justify each step.

(a) $\lim_{x \to a} x^n$, where n is a positive integer.

(b) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(c) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Remark. What happens if we just plugged in the numbers in the previous example?

Example. Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$. Conclude that direct substitution doesn't always work.

Question. Why does "factor and cancel" work?

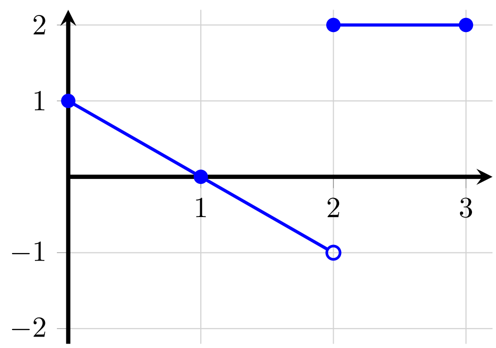
Example. Evaluate $\lim_{h\to 0} \dfrac{(3+h)^2 - 9}{h}$ by simplifying the function.

Example. Evaluate $\lim_{t\to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2}$ by rationalizing.

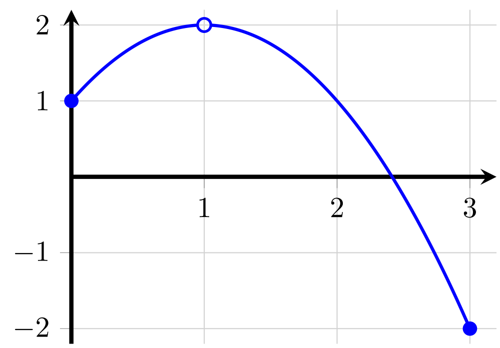
Example. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Example. Use the graphs of f and g to find $\lim_{x \to 1} f(g(x))$.

Graph of f



Graph of g

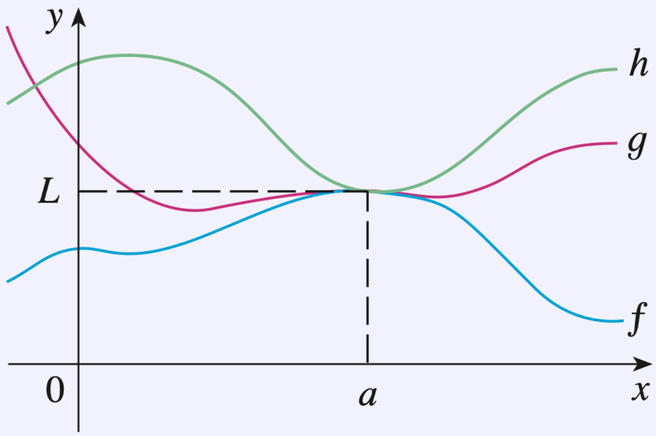


Theorem (Squeeze Theorem)**.** If $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$



Example. Show that $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$.

Remark. Below are the graphs of $y = x^2$, $y = -x^2$, and $y = x^2 \sin \frac{1}{x}$. The squeeze theorem applies.