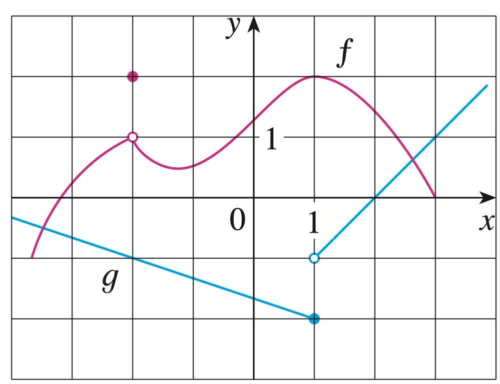
2.3 Calculating Limits Using the Limit Laws

Theorem. Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

1. $\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. Sum Law
2. $\lim_{x\to a}\big(f(x)-g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. Difference Law
3. $\lim_{x\to a}\big(c\,f(x)\big) = c\lim_{x\to a} f(x)$. Constant Multiple Law
4. $\lim_{x\to a}\big(f(x)\,g(x)\big) = \Big(\lim_{x\to a} f(x)\Big)\cdot\Big(\lim_{x\to a} g(x)\Big)$. Product Law
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$. Quotient Law

Remark. All of these hold for one-sided limits as well.

Example. Use the Limit Laws and the graphs of f and g to evaluate the following limits.



- $\lim_{x\to -2}[f(x)+5g(x)]$

[Sum Law]

$$= \lim_{x\to -2} f(x) + \lim_{x\to -2} 5g(x)$$

[Constant Multiple Law]

$$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$$

$$= 1 + 5(-1)$$

$$= -4$$

- $\lim_{x\to 1}[f(x)g(x)]$

Q: Why can't we say $\lim_{x\to 1}[f(x)\cdot g(x)] = \lim_{x\to 1} f(x)\cdot \lim_{x\to 1} g(x)$

A: $\lim_{x\to 1} g(x)$ does not exist.
Must use one-sided limits

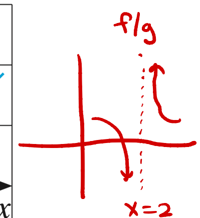
$$\lim_{x\to 1^-}[f(x)\cdot g(x)] = \lim_{x\to 1^-} f(x)\cdot \lim_{x\to 1^-} g(x) = 2\cdot -2 = -4$$

$$\lim_{x\to 1^+}[f(x)\cdot g(x)] = \lim_{x\to 1^+} f(x)\cdot \lim_{x\to 1^+} g(x) = 2\cdot -1 = -2$$

$$\boxed{\lim_{x\to 1}[f(x)\cdot g(x)] = \text{D.N.E.}}$$

since the left and right hand limits are not equal.

- $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$



We can't say $\lim_{x\to 2}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)}$ because $\lim_{x\to 2} g(x) = 0$

Analyze $\dfrac{f(x)}{g(x)}$ as a function

As $x\to 2^-$, it will be $\dfrac{\text{pos}\#}{\text{small neg}\#} \to -\infty$

As $x\to 2^+$, it will be $\dfrac{\text{pos}\#}{\text{small pos}\#} \to \infty$

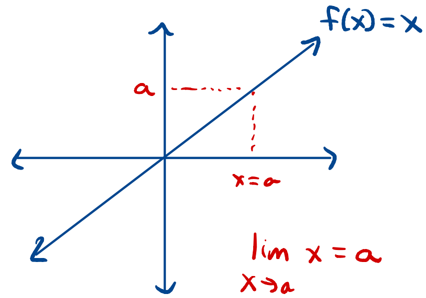
Because these are different $\lim_{x\to 2}\dfrac{f(x)}{g(x)} = \text{DNE}$

Theorem. Suppose the limit $\lim_{x\to a} f(x)$ exists. Then:

1. $\lim_{x\to a} \left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$ — Power Law
2. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ — Root Law

Example. Evaluate the following limits and justify each step.

(a) $\lim_{x\to a} x^n$, where n is a positive integer.



$\lim_{x\to a} x = a$

$$\lim_{x\to a} x^n \underset{\text{Power Law}}{=} \left[\lim_{x\to a} x\right]^n \underset{\lim_{x\to a} x = a}{=} a^n$$

(b) $\lim_{x\to 5}(2x^2 - 3x + 4)$

$= \lim_{x\to 5} 2x^2 - \lim_{x\to 5} 3x + \lim_{x\to 5} 4$ [Sum/difference Law]

$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$ [Constant multiple law]

$= 2(5)^2 - 3(5) + 4$ $\left[\lim_{x\to a} x^n = a^n\right]$

$= 50 - 15 + 4 = 39$

(c) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Top: $\lim_{x\to -2} x^3 + 2x^2 - 1 = \lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1 = (-2)^3 + 2\cdot(-2)^2 - 1$
$= -8 + 8 - 1 = -1$

Bottom: $\lim_{x\to -2} 5 - 3x = \lim_{x\to -2} 5 - 3\lim_{x\to -2} x = 5 - 3(-2) = 5 + 6 = 11$

By the Quotient Law, $\boxed{\lim_{x\to -2} \dfrac{x^3+2x^2-1}{5-3x} = \dfrac{-1}{11}}$

These are the same as before! $\rightarrow$ They are the same because the functions are continuous at those values

Remark. What happens if we just plugged in the numbers in the previous example?

$$2\cdot(5)^2 - 3(5) + 4 = 39 \quad \text{and} \quad \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \frac{-1}{11}$$

Quiz

Example. Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$. Conclude that direct substitution doesn't always work.

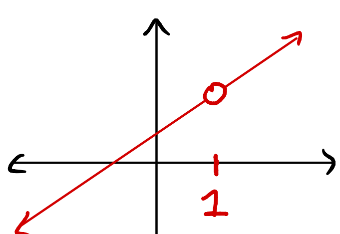
When we plug in $x=1$, we get $\dfrac{1^2-1}{1-1} = \dfrac{0}{0}$ ✗

Instead,
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x\to 1} x+1$$
$$= \lim_{x\to 1} x + \lim_{x\to 1} 1$$
$$= 1 + 1$$
$$= 2$$

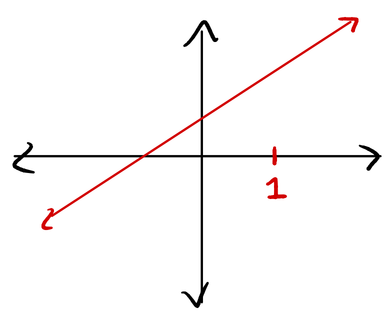
✱ This is called "Factor and Cancel"

Question. Why does "factor and cancel" work?

It works because the functions $\dfrac{x^2-1}{x-1}$ and $x+1$ are the same everywhere besides $x=1$ and we don't consider $x=1$ when determining the limit



$\dfrac{x^2-1}{x-1}$



$x+1$

EVALUATING LIMITS

- TABLES
  - Estimate
- GRAPHS
  - Bugs
- ALGEBRA
  - Limit Laws
  - Factor & Cancel
  - Rationalizing
  - Squeeze Theorem

Example. Evaluate $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$ by simplifying the function.

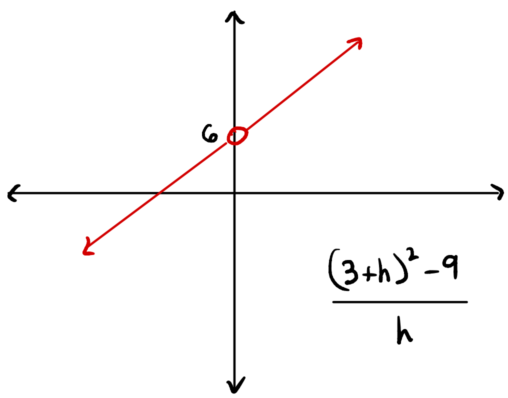
Cannot use quotient property because $\frac{0}{0}$ ✗

$$\lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6+h$$

$$= \lim_{h\to 0} 6 + \lim_{h\to 0} h$$

$$= \boxed{6}$$



Example. Evaluate $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$ by rationalizing.

The quotient property would give $\frac{0}{0}$ ✗

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right)$$

← This is 1

$\sqrt{t^2+9}+3$ is called the conjugate

$$= \lim_{t\to 0} \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \boxed{\frac{1}{6}}$$

Quotient Property, Root Property, Sum Property, etc...

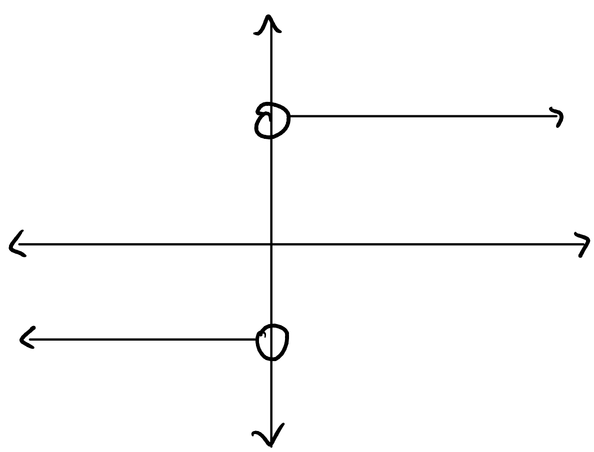
Example. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Look at left and right hand limits separately.

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

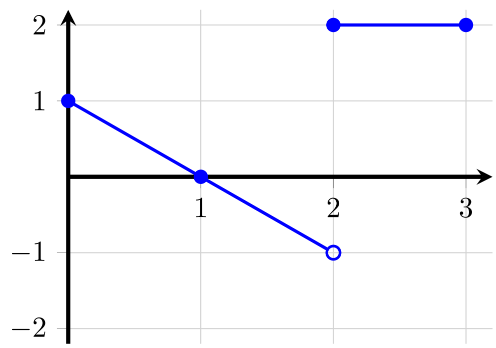
These are different, so the limit does not exist.



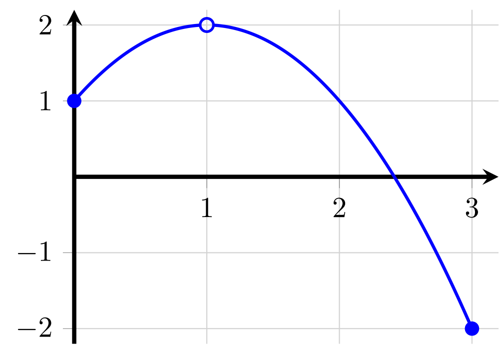
$\frac{|x|}{x}$

Example. Use the graphs of f and g to find $\lim_{x\to 1} f(g(x))$.

Graph of f



Graph of g



Idea: look at one-sided limits and use substitution

$\lim_{x\to 1^-} f(g(x))$

Set $y = g(x)$

Key: As $x \to 1^-$, $y \to 2^-$

$\Rightarrow \lim_{x\to 1^-} f(g(x)) = \lim_{y\to 2^-} f(y)$

$= -1$

$\lim_{x\to 1^+} f(g(x))$

Set $y = g(x)$

As $x \to 1^+$, $y \to 2^-$

$\Rightarrow \lim_{x\to 1^+} f(g(x)) = \lim_{y\to 2^-} f(y)$

$= -1$

In both cases, y approaches 2 from values <u>below</u> 2

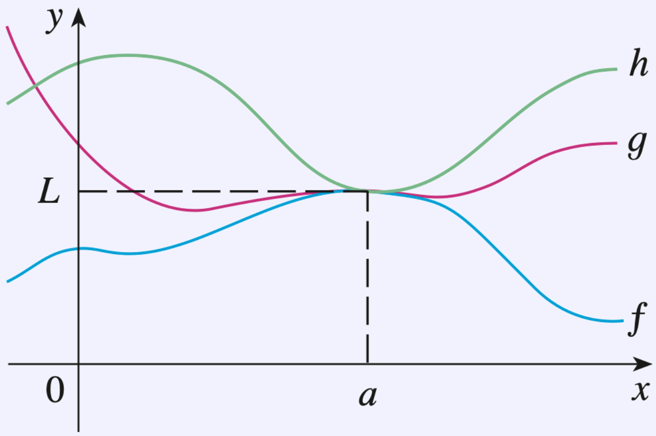
$$\boxed{\lim_{x\to 1} f(g(x)) = -1}$$

Theorem (Squeeze Theorem)**.** If $f(x) \le g(x) \le h(x)$ for x near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$



Example. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$

this is D.N.E.

- We can't use the product property: $\lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$
- Can we bound $x^2\sin\left(\frac{1}{x}\right)$ between two functions?

$$-1 \le \sin\left(\tfrac{1}{x}\right) \le 1$$

$$-x^2 \le x^2\sin\left(\tfrac{1}{x}\right) \le x^2$$

Multiply through by x^2. x^2 is always ≥ 0 so this won't affect the chain of inequalities.

- Taking limits,

$$\underbrace{\lim_{x\to 0} -x^2}_{=0} \le \lim_{x\to 0} x^2\sin\left(\tfrac{1}{x}\right) \le \underbrace{\lim_{x\to 0} x^2}_{=0}$$

- By the Squeeze Thm, $\lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0$

Remark. Below are the graphs of $y = x^2$, $y = -x^2$, and $y = x^2 \sin \frac{1}{x}$. The squeeze theorem applies.