2.2 The Limit of a Function

Definition. Let f be defined for all x close to a (except possibly at a). We say

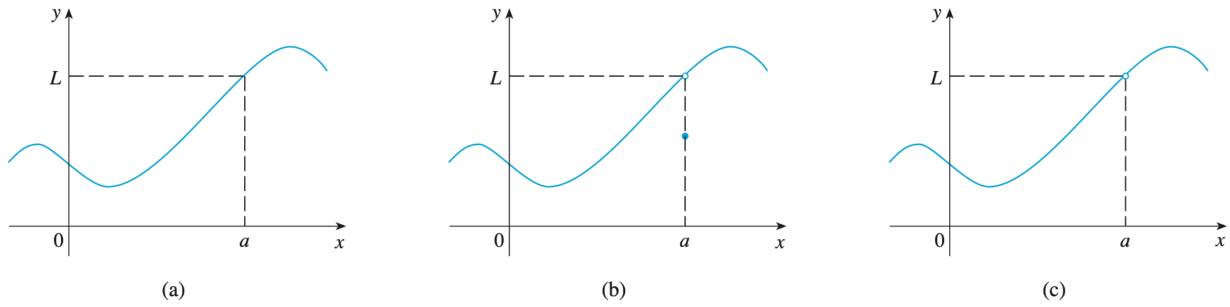
$$\lim_{x \to a} f(x) = L$$

if the values $f(x)$ can be made as close to L as we like by taking x sufficiently close to a (with $x \neq a$). In other words, as x approaches a, $f(x)$ approaches L.

Note: The limit describes the behavior of $f(x)$ near a, not necessarily its value at a.

Example. Using a table, find $\lim_{x \to 0} \left(x^3 + \dfrac{\cos(5x)}{10,000} \right)$

Example. Understand why $\lim_{x \to a} f(x) = L$ in all three cases below.



Example. The Heaviside function H is defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Explain why $\lim_{t \to 0} H(t)$ does not exist.

Definition. Let f be defined for x-values near a.

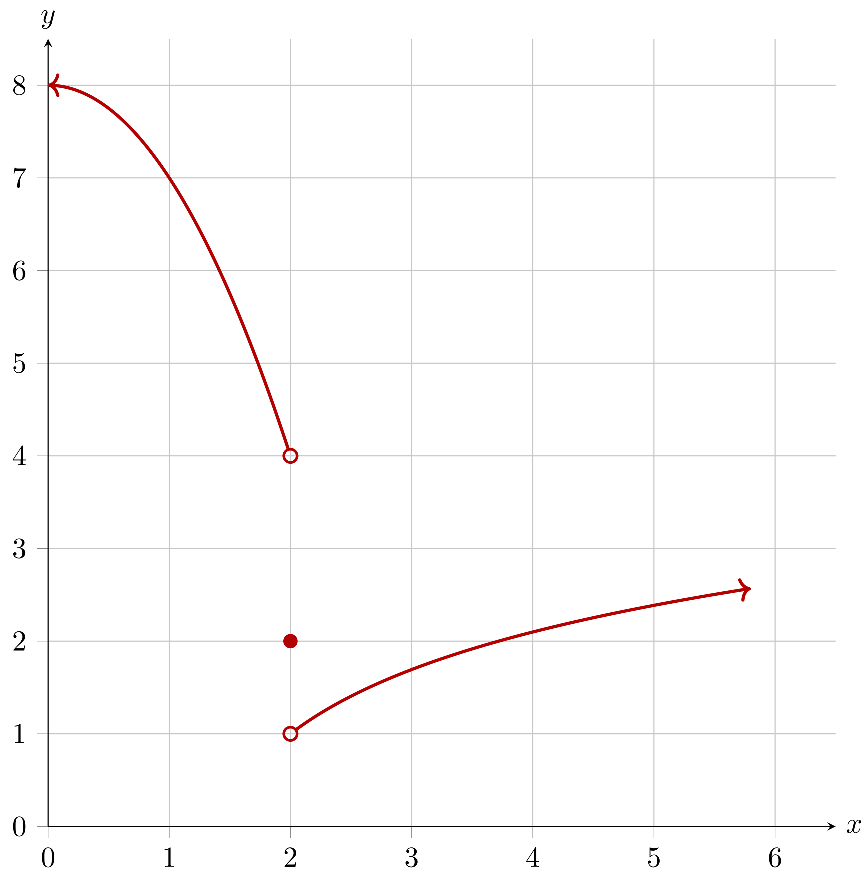
- **Right-hand limit.** We write $\lim_{x \to a^+} f(x) = L$ if the outputs $f(x)$ approach L as x approaches a, for x-values *greater than* a.
- **Left-hand limit.** We write $\lim_{x \to a^-} f(x) = L$ if the outputs $f(x)$ approach L as x approaches a, for x-values *less than* a.

Theorem. For the two-sided limit $\lim_{x \to a} f(x)$ to be equal to L, both one-sided limits must also be equal to L:

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

The two-sided limit $\lim_{x \to a} f(x)$ does not exist if the left and right limits are not both defined and equal.

Example. The graph of $f(x)$ is shown below. Compute the following:



- $\lim_{x \to 2^-} f(x) =$

- $\lim_{x \to 2^+} f(x) =$

- $\lim_{x \to 2} f(x) =$

- $f(2) =$

Example. Find $\lim_{x \to 0} \left(\dfrac{1}{x^2} \right)$ if it exists.

Definition. Let f be defined for x-values on both sides of a (except possibly at a itself).

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made *arbitrarily large* by taking x sufficiently close to a (with $x \neq a$). Similarly,

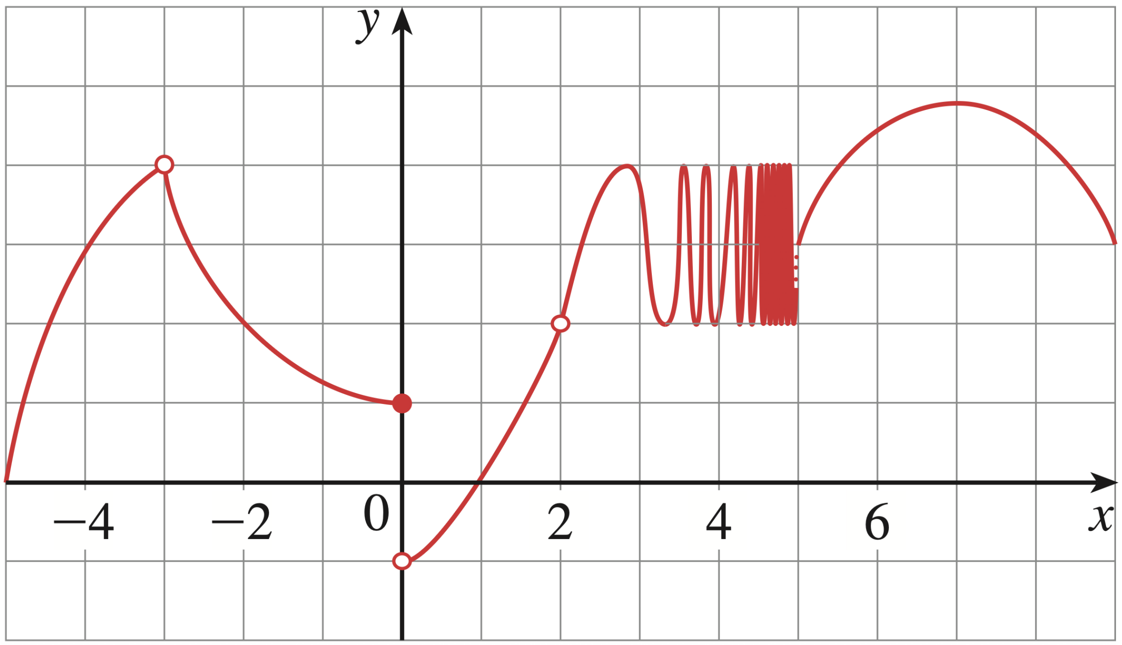
$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made *arbitrarily negative* by taking x sufficiently close to a.

Remark. If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$), the limit **does not exist**. The symbol ∞ specifies *how* it fails to exist. For a two-sided infinite limit, both one-sided limits must blow up the same way:

$$\lim_{x \to a^-} f(x) = \infty \text{ and } \lim_{x \to a^+} f(x) = \infty \quad \text{(or both } = -\infty\text{)}.$$

Example. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x \to -3^-} h(x)$

(b) $\lim_{x \to -3^+} h(x)$

(c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$

(e) $\lim_{x \to 0^-} h(x)$

(f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$

(h) $h(0)$

(i) $\lim_{x \to 2} h(x)$

(j) $h(2)$

(k) $\lim_{x \to 5^+} h(x)$

(l) $\lim_{x \to 5^-} h(x)$

Example. Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x \to 0^-} f(x) = -1$
- $\lim_{x \to 0^+} f(x) = 2$
- $f(0) = 1$

Example. Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 3^-} f(x) = -2$
- $\lim_{x \to 3^+} f(x) = 2$
- $f(0) = -1$
- $f(3) = 1$