Yesterday: we computed instantaneous change by computing average change over smaller and smaller intervals.

Issue:

1. It takes time to compute average change over all these intervals
2. How do we actually know for sure what these values approach?

New goal: make this “limiting procedure” precise.

2.2 The Limit of a Function

Definition. Let f be defined for all x close to a (except possibly at a). We say

$$\lim_{x \to a} f(x) = L$$

if the values $f(x)$ can be made as close to L as we like by taking x sufficiently close to a (with $x \neq a$). In other words, as x approaches a, $f(x)$ approaches L.

Note: The limit describes the behavior of $f(x)$ near a, not necessarily its value at a.

The method we used yesterday

Example. Using a table, find $\lim_{x \to 0} \left(x^3 + \frac{\cos(5x)}{10,000} \right)$

x	$x^3 + \frac{\cos(5x)}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.001	0.0000999
-0.001	0.0000999

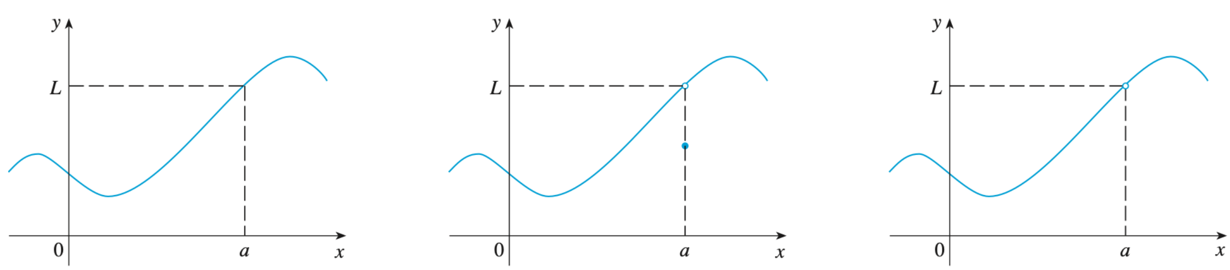
$1^3 + \frac{\cos(5\cdot 1)}{10{,}000}$

$(0.5)^3 + \frac{\cos(5\cdot 0.5)}{10{,}000}$

By examining our table, the values approach 0.0001 as x approaches 0:

$$\boxed{\lim_{x \to 0} \left(x^3 + \frac{\cos(5x)}{10000} \right) = 0.0001}$$

Example. Understand why $\lim_{x \to a} f(x) = L$ in all three cases below.



(a) (b) (c)

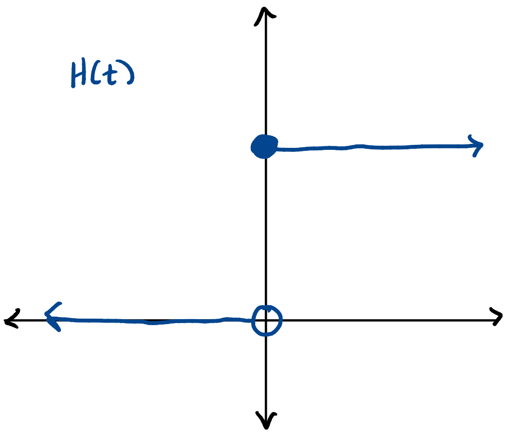
Important: when computing $\lim_{x \to a} f(x)$ we look at $f(x)$ around $x=a$. What happens at $x=a$ doesn't matter.

In each case, we can make $f(x)$ as close as we want to L by getting close enough to $x=a$.

Example. The Heaviside function H is defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Explain why $\lim_{t \to 0} H(t)$ does not exist.



As $t \to 0$, there is no single number that $H(t)$ approaches.

- As t approaches 0 from the left, $H(t)$ approaches 0
- As t approaches 0 from the right $H(t)$ approaches 1

$\Rightarrow$ In this case, $\boxed{\lim_{t \to 0} H(t) = \text{DNE}}$

Definition. Let f be defined for x-values near a.

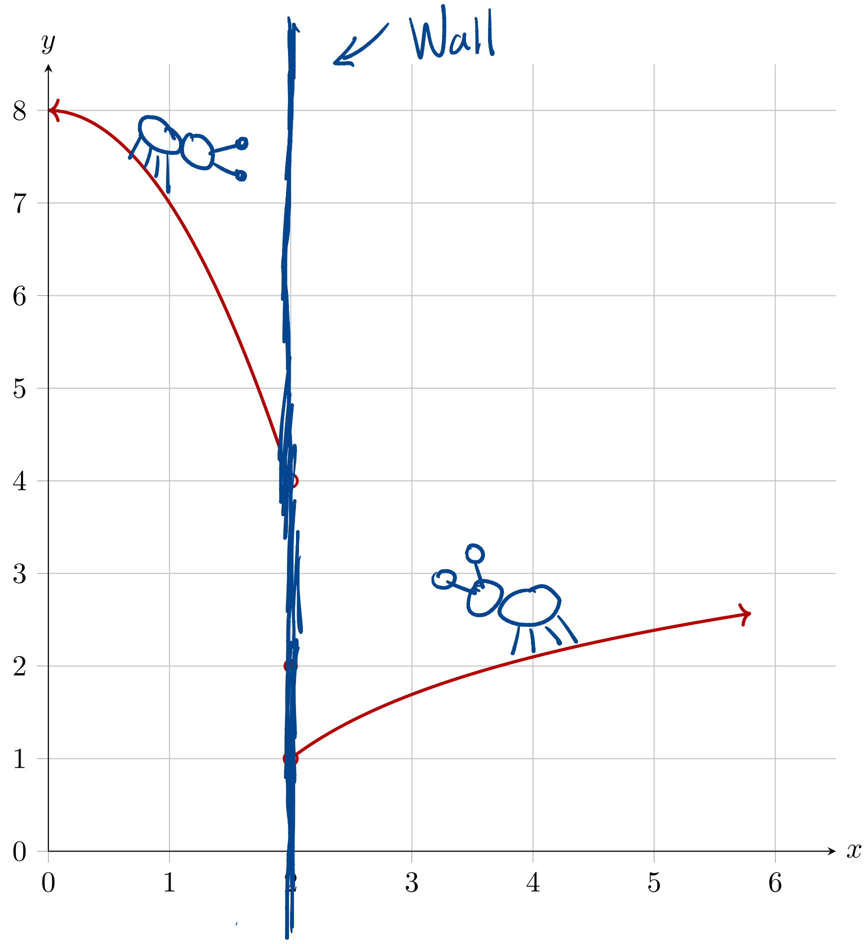
- **Right-hand limit.** We write $\lim_{x \to a^+} f(x) = L$ if the outputs $f(x)$ approach L as x approaches a, for x-values *greater than* a.
- **Left-hand limit.** We write $\lim_{x \to a^-} f(x) = L$ if the outputs $f(x)$ approach L as x approaches a, for x-values *less than* a.

Theorem. For the two-sided limit $\lim_{x \to a} f(x)$ to be equal to L, both one-sided limits must also be equal to L:

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

The two-sided limit $\lim_{x \to a} f(x)$ does not exist if the left and right limits are not both defined and equal.

Example. The graph of $f(x)$ is shown below. Compute the following:



Where will the bug hit the wall?

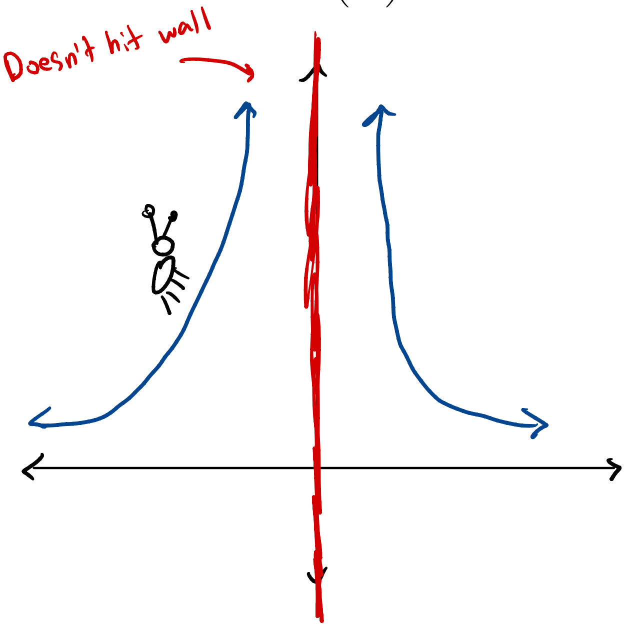
- $\lim\limits_{x \to 2^-} f(x) =$ 4

- $\lim\limits_{x \to 2^+} f(x) =$ 1

- $\lim\limits_{x \to 2} f(x) =$ DNE

- $f(2) =$ 2

Example. Find $\lim_{x \to 0} \left(\frac{1}{x^2} \right)$ if it exists.



Both the left and right hand limits go off to infinity

$$\boxed{\lim_{x \to 0} f(x) = \text{D.N.E.}}$$

$= \infty$

"If the limit never approaches anything... the limit does not exist"

– Cady Heron

Definition. Let f be defined for x-values on both sides of a (except possibly at a itself).

$$\lim_{x \to a} f(x) = \infty$$

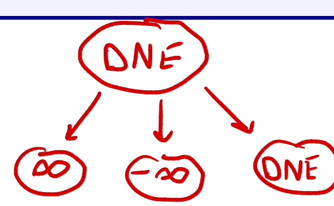
means that the values of $f(x)$ can be made *arbitrarily large* by taking x sufficiently close to a (with $x \neq a$). Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

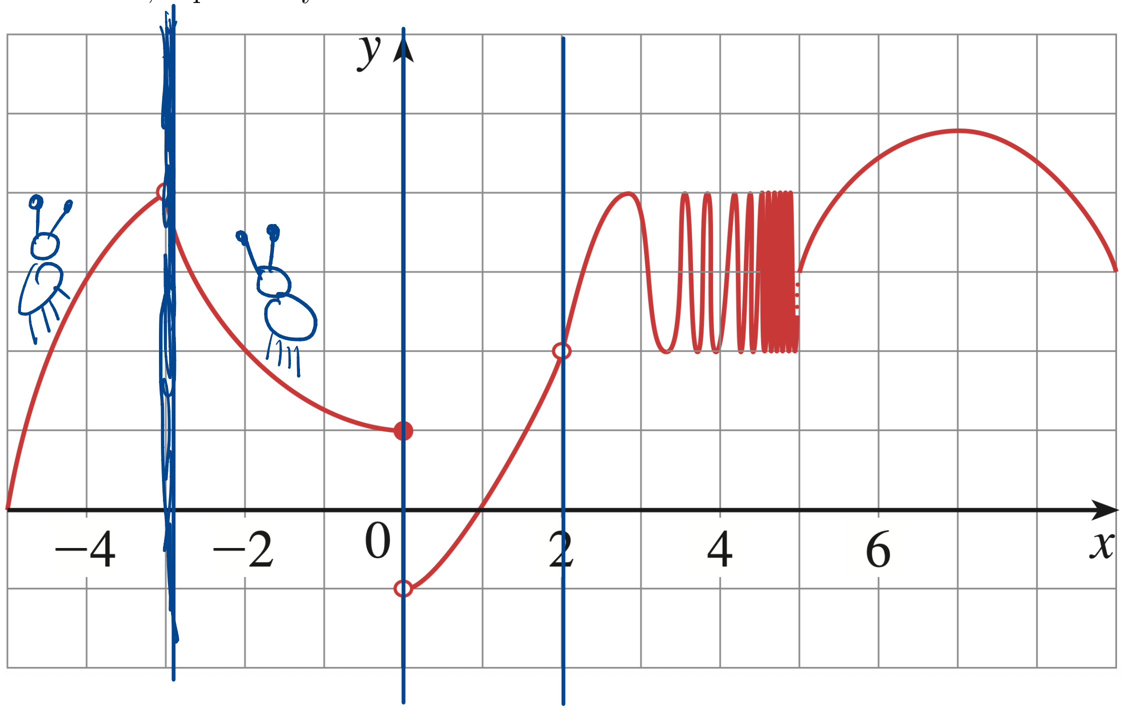
means that the values of $f(x)$ can be made *arbitrarily negative* by taking x sufficiently close to a.

Remark. If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$), the limit **does not exist**. The symbol ∞ specifies *how* it fails to exist. For a two-sided infinite limit, both one-sided limits must blow up the same way:

$$\lim_{x \to a^-} f(x) = \infty \text{ and } \lim_{x \to a^+} f(x) = \infty \quad \text{(or both } = -\infty\text{)}.$$



Example. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x\to -3^-} h(x) = 4$

(b) $\lim_{x\to -3^+} h(x) = 4$

(c) $\lim_{x\to -3} h(x) = 4$

(d) $h(-3)$ = Undefined

(e) $\lim_{x\to 0^-} h(x) = 1$

(f) $\lim_{x\to 0^+} h(x) = -1$

(g) $\lim_{x\to 0} h(x)$ = DNE

(h) $h(0) = 1$

(i) $\lim_{x\to 2} h(x) = 2$

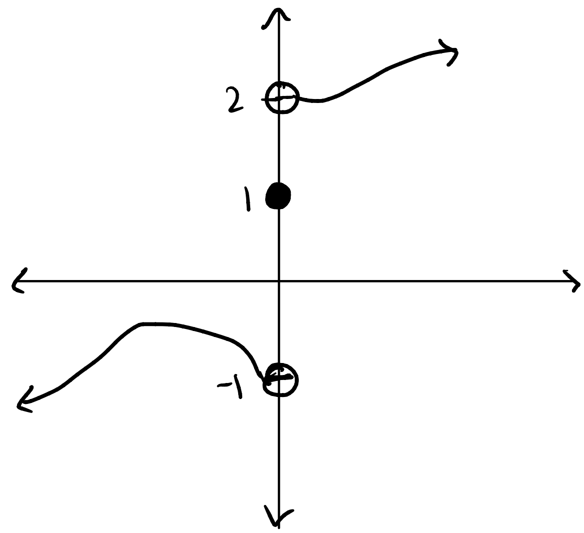
(j) $h(2)$ = undefined

(k) $\lim_{x\to 5^+} h(x) = 3$

(l) $\lim_{x\to 5^-} h(x)$ = DNE

Example. Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x \to 0^-} f(x) = -1$
- $\lim_{x \to 0^+} f(x) = 2$
- $f(0) = 1$



Example. Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 3^-} f(x) = -2$
- $\lim_{x \to 3^+} f(x) = 2$
- $f(0) = -1$
- $f(3) = 1$