Strategy: From precalc, we can compute the average rate of change of a function over an interval by computing $\frac{\Delta y}{\Delta x}$. Let's approximate instantaneous velocity using the average velocity on an extremely small interval.

2.1 The tangent and velocity problems

Example. Galileo found that the distance fallen by any freely falling body (neglecting air resistance) after t seconds is $s(t) = 4.9t^2$ meters. Suppose that a ball is dropped from 450m above the ground. Our goal is to find the velocity of the ball after *exactly* 5 seconds.

Question. How do you compute the *average velocity* over a time interval $[a, b]$?

$$\text{Average Velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b) - s(a)}{b-a}$$

Question. Find the average velocity of the ball over the given time intervals.

Time Interval	Average Velocity
$5 \le t \le 6$	53.9 m/s
$5 \le t \le 5.1$	49.49 m/s
$5 \le t \le 5.05$	49.245 m/s
$5 \le t \le 5.01$	49.049 m/s
$5 \le t \le 5.001$	49.0049 m/s

$$\frac{s(6)-s(5)}{6-5} = \frac{4.9(6)^2 - 4.9(5)^2}{1} = 53.9 \text{ m/s}$$

$$\frac{s(5.1)-s(5)}{5.1-5} = \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}$$

$$\frac{s(5.05)-s(5)}{5.05-5} = \frac{4.9(5.05)^2 - 4.9(5)^2}{0.05} = 49.245 \text{ m/s}$$

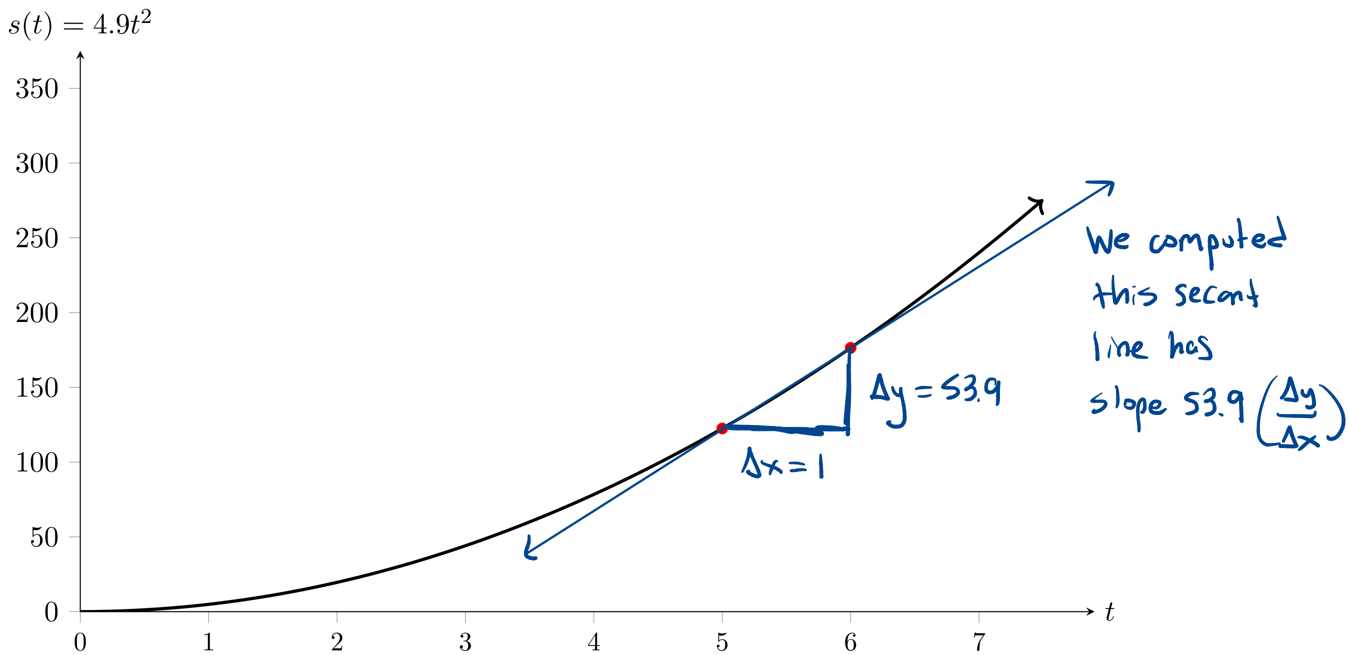
$$\frac{s(5.01)-s(5)}{5.01-5} = \frac{4.9(5.01)^2 - 4.9(5)^2}{0.01} = 49.049 \text{ m/s}$$

$$\frac{s(5.001)-s(5)}{5.001-5} = \frac{4.9(5.001)^2 - 4.9(5)^2}{0.001} = 49.0049 \text{ m/s}$$

Question. Make a guess for what the velocity of the ball will be after *exactly* 5 seconds.

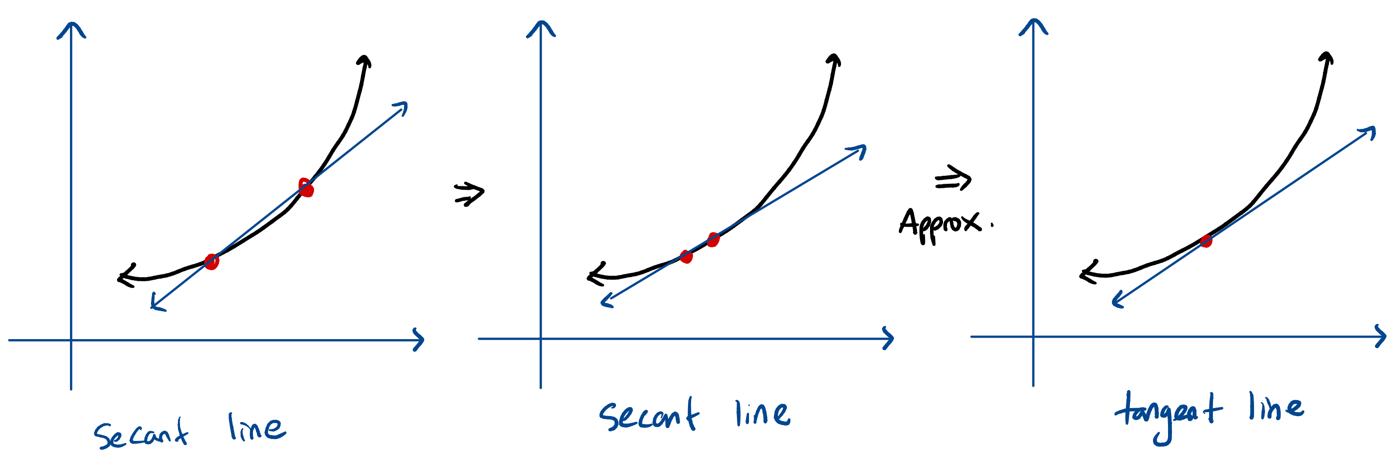
- As we make these intervals smaller and smaller, these values approach 49 m/s
- We will eventually prove using Calculus that the ball is falling exactly 49 m/s at $t = 5$ sec.

Question. Explain what is happening graphically in the above scenario.



Key: Average velocities over an interval correspond to slopes of secant lines (lines hitting the graph at 2 points)

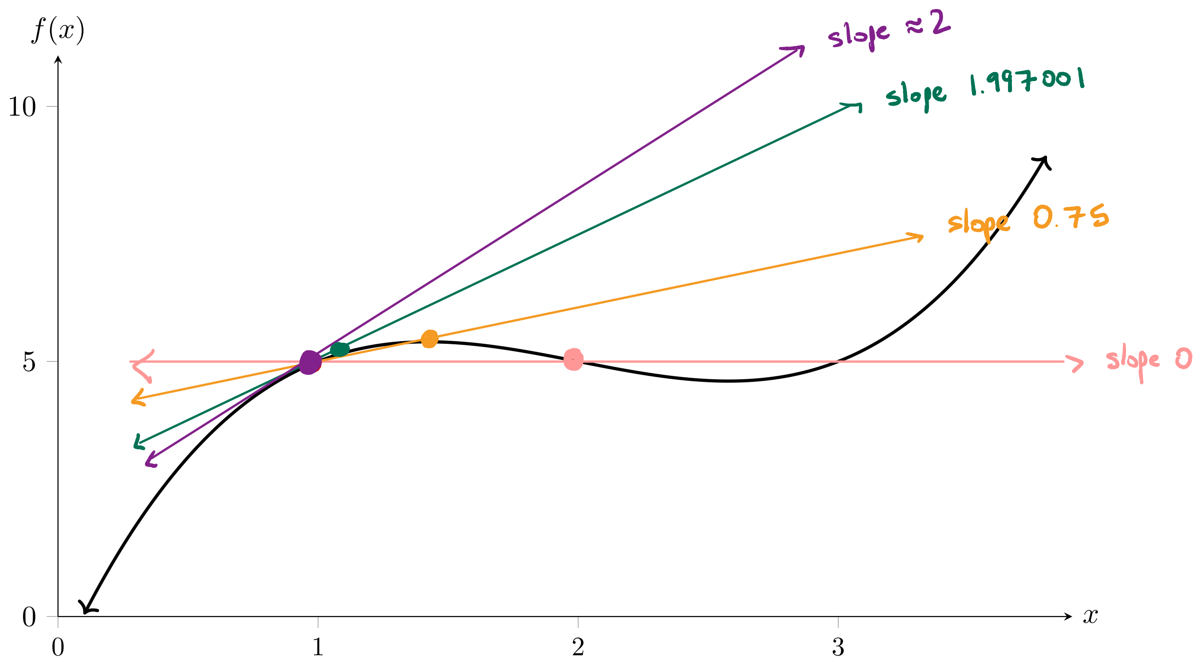
We took the second point (at $t=6$) and made it closer and closer to $t=5$, shrinking the interval to 0.



Big idea: the slope of the tangent line (a line that just touches the curve) is the instantaneous velocity.

2.1 The tangent and velocity problems (activity)

Below is the graph of the function $f(x) = (x-2)^3 - x + 7$. The goal of this activity is to estimate the instantaneous change of $f(x)$ at $x = 1$.



Question 1. Find the average rate of change of $f(x)$ on the interval $[1, 2]$. Draw the corresponding secant line on the graph above.

$$\frac{f(2)-f(1)}{2-1} = \frac{5-5}{1} = 0$$

Question 2. Find the average rate of change of $f(x)$ on the interval $[1, 1.5]$. Draw the corresponding secant line on the graph above.

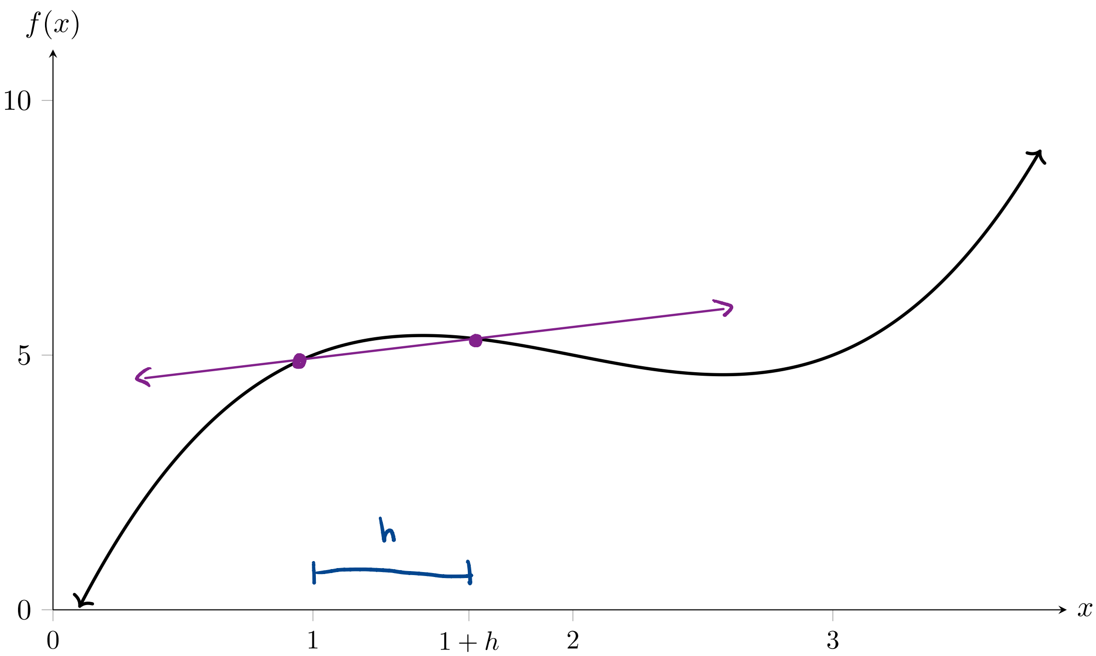
$$\frac{f(1.5)-f(1)}{1.5-1} = \frac{0.375}{0.5} = 0.75$$

Question 3. Find the average rate of change of $f(x)$ on the interval $[1, 1.001]$. Draw the corresponding secant line on the graph above.

$$\frac{f(1.001)-f(1)}{1.001-1} = \frac{0.001997001}{0.001} = 1.997001$$

Question 4. Estimate the instantaneous change at $x = 1$. Draw the corresponding tangent line on the graph above.

We estimate the instantaneous change to be approximately 2



Question 5. Let h be an arbitrary number that represents the distance between $x = 1$ and another x value. Find the average rate of change of $f(x)$ between $x = 1$ and $x = 1 + h$.

$$\frac{f(1+h) - f(1)}{1+h-1} = \frac{f(1+h) - f(1)}{h}$$

★ Difference Quotient

Question 6. Explain what we need to do to h in the expression above to estimate the instantaneous change at $x = 1$.

We need to compute $\frac{f(1+h) - f(1)}{h}$ for different values of h as h gets closer and closer to 0.

Question 7. Write down an expression that computes the instantaneous rate of change at an arbitrary value $x = a$.

$\frac{f(a+h) - f(a)}{h}$ computes the average rate of change between $x=a$ and $x=a+h$ and we need to compute this as h gets smaller and smaller.

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$