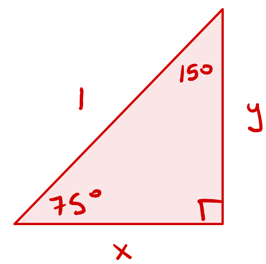
IC 27: Sum and Difference of Angles

Use a sum or difference formula to evaluate each expression.

1. $\sin(75°)$

$$\begin{aligned}\sin(75°) &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$



2. $\cos(15°)$

$\cos(15°) = \sin(75°)$

$$\begin{aligned}\cos(15°) &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

3. $\tan(105°)$

$$\begin{aligned}\tan(105°) &= \tan(60° + 45°)\\ &= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}}\end{aligned}$$

optional simplification

$$\left\{\begin{aligned} &= \frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}\\ &= \frac{(1+\sqrt{3})^2}{1-3}\\ &= \frac{4+2\sqrt{3}}{-2}\\ &= -2-\sqrt{3}\end{aligned}\right.$$

4. $\cos\left(\frac{13\pi}{12}\right)$

$$\begin{aligned}\cos\left(\frac{13\pi}{12}\right) &= \cos\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)\\ &= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= -\frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

5. $\tan\left(\frac{7\pi}{12}\right)$

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) &= \tan\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)\\ &= \frac{\tan\left(\frac{3\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}\\ &= \frac{-1-\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}-1}{\sqrt{3}-1} = \frac{-(\sqrt{3}+1)}{-(-\sqrt{3}+1)}\\ &= \frac{1+\sqrt{3}}{1-\sqrt{3}}\end{aligned}$$

Write each expression as a single trig function and evaluate exactly.

7. $\sin(12^\circ)\cos(18^\circ)+\cos(12^\circ)\sin(18^\circ)$

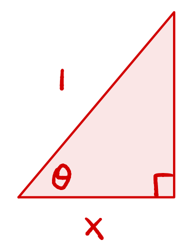
$$\sin A\cos B+\cos A\sin B = \sin(A+B)$$
$$\Rightarrow \sin(12^\circ)\cos(18^\circ)+\cos(12^\circ)\sin(18^\circ) = \sin(30^\circ) = \frac{1}{2}$$

8. $\cos(70^\circ)\cos(20^\circ)-\sin(70^\circ)\sin(20^\circ)$

$$\cos A\cos B-\sin A\sin B = \cos(A+B)$$
$$\Rightarrow \cos(70^\circ)\cos(20^\circ)-\sin(70^\circ)\sin(20^\circ) = \cos(90^\circ) = 0$$

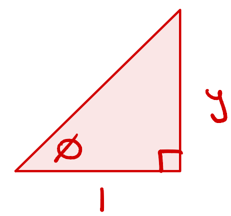
9. Show that $\sin(x-\pi) = -\sin(x)$

$$\begin{aligned}\sin(x-\pi) &= \sin x\cos\pi - \cos x \sin\pi \\ &= \sin x\cdot(-1) - \cos x\cdot 0 \\ &= -\sin x\end{aligned}$$



10. Assume $x, y > 0$. Find $\cos\left(\cos^{-1}(x) - \tan^{-1}(y)\right)$ in terms of x and y.
Let $\theta = \cos^{-1}(x), \phi = \tan^{-1}(y)$. Then:

$$\begin{aligned}\cos\theta = x \quad &\Rightarrow \sin\theta = \sqrt{1-x^2} \\ \tan\phi = y \quad &\Rightarrow \sin\phi = \frac{y}{\sqrt{1+y^2}}, \quad \cos\phi = \frac{1}{\sqrt{1+y^2}}\end{aligned}$$



$$\begin{aligned}\cos(\theta-\phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi \\ &= x\cdot\frac{1}{\sqrt{1+y^2}} + \sqrt{1-x^2}\cdot\frac{y}{\sqrt{1+y^2}} \\ &= \frac{x + y\sqrt{1-x^2}}{\sqrt{1+y^2}}\end{aligned}$$

11. Let $\cos\alpha = \dfrac{5}{13}$ with $0 \le \alpha \le \frac{\pi}{2}$, and $\sin\beta = \dfrac{3}{5}$ with $0 \le \beta \le \frac{\pi}{2}$. Find the exact value of $\tan(\alpha+\beta)$.

$$\begin{aligned}\sin\alpha &= \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}, \quad \cos\beta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5} \\ \tan\alpha &= \frac{12}{5}, \quad \tan\beta = \frac{3}{4} \\ \tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}} \\ &= \frac{\frac{63}{20}}{1-\frac{36}{20}} = \frac{63}{20}\cdot\frac{20}{-16} = -\frac{63}{16}\end{aligned}$$