

Math 1150: Final Exam Practice

Radical & Exponential Simplification

1. Simplify the expression. Assume all variables are positive

$$\left(\frac{81a^6}{b^{12}}\right)^{1/2}.$$

Solution:

$$\sqrt{\frac{81a^6}{b^{12}}} = \frac{\sqrt{81} \sqrt{a^6}}{\sqrt{b^{12}}} = \frac{9a^3}{b^6}.$$

2. Simplify

$$\left(\frac{27p^3q^{-6}}{3p^{-2}q^2}\right)^{1/3}.$$

Solution: First simplify inside the cube root:

$$\frac{27p^3q^{-6}}{3p^{-2}q^2} = 9p^{3-(-2)}q^{-6-2} = 9p^5q^{-8}.$$

Then take the one-third power:

$$(9p^5q^{-8})^{1/3} = 9^{1/3}p^{5/3}q^{-8/3} = \sqrt[3]{9}p^{5/3}q^{-8/3}.$$

Inequalities

1. Solve the inequality $|3x - 4| > 5$.

Solution:

$$3x - 4 > 5 \implies 3x > 9 \implies x > 3,$$

or

$$3x - 4 < -5 \implies 3x < -1 \implies x < -\frac{1}{3}.$$

Final answer: $(-\infty, -\frac{1}{3}) \cup (3, \infty)$.

2. Solve the inequality $|-4x + 8| \leq 12$.

Solution:

$$-12 \leq -4x + 8 \leq 12 \implies -20 \leq -4x \leq 4 \implies 5 \geq x \geq -1.$$

Final answer: $[-1, 5]$.

3. Solve $(x + 1)(x - 7) < 0$.

Solution: The roots are -1 and 7 . Using a sign chart, we obtain:

$$-1 < x < 7.$$

4. Solve $(x - 2)(x + 5) < 0$.

Solution: The roots are 2 and -5 . Using a sign chart, we obtain

$$-5 < x < 2.$$

Difference Quotients

1. Compute the difference quotient $\frac{f(x+h) - f(x)}{h}$ for $f(x) = 2x^2 + 5x$.

Solution:

$$f(x+h) - f(x) = 2(x+h)^2 + 5(x+h) - (2x^2 + 5x) = 4xh + 2h^2 + 5h,$$

so

$$\frac{f(x+h) - f(x)}{h} = 4x + 2h + 5.$$

2. Compute the difference quotient $\frac{f(x+h) - f(x)}{h}$ for $f(x) = 3x^2 - 5x + 1$.

Solution:

$$f(x+h) - f(x) = 3(x+h)^2 - 5(x+h) + 1 - (3x^2 - 5x + 1) = 6xh + 3h^2 - 5h,$$

thus

$$\frac{f(x+h) - f(x)}{h} = 6x + 3h - 5.$$

Polynomial Equations

1. Solve for x : $(x - 4)(x + 6) = 5$.

Solution:

$$(x - 4)(x + 6) = 5 \implies x^2 + 2x - 24 = 5 \implies x^2 + 2x - 29 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 116}}{2} = \frac{-2 \pm \sqrt{120}}{2} = -1 \pm \sqrt{30}.$$

2. Solve for x : $(2x - 3)(x + 4) = 6$.

Solution:

$$(2x - 3)(x + 4) = 6 \implies 2x^2 + 5x - 12 = 6 \implies 2x^2 + 5x - 18 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 144}}{4} = \frac{-5 \pm 13}{4} \implies x = 2 \quad \text{or} \quad x = -\frac{9}{2}.$$

Domain & Rational Functions

1. Determine the domain of $f(x) = \frac{\sqrt{x+1}}{x^2-9}$.

Solution: Require $x + 1 \geq 0$ and $x^2 - 9 \neq 0$:

$$x \geq -1, \quad x \neq \pm 3.$$

Final answer: $[-1, 3) \cup (3, \infty)$.

2. Determine the domain of $f(x) = \frac{\sqrt{5x+4}}{x^2-1}$.

Solution: Require $5x + 4 \geq 0$ and $x^2 - 1 \neq 0$:

$$x \geq -\frac{4}{5}, \quad x \neq \pm 1.$$

Since $-1 < -\frac{4}{5}$, only $x = 1$ must be excluded. Thus $[-\frac{4}{5}, 1) \cup (1, \infty)$.

3. Identify all vertical asymptotes of $f(x) = \frac{\sqrt{x-3}}{x^2-4x+3}$.

Solution: Domain requires $x - 3 \geq 0$, so $x \geq 3$. Factor the denominator:

$$x^2 - 4x + 3 = (x - 1)(x - 3).$$

Within $x \geq 3$, only $x = 3$ makes the denominator zero. Hence there is a vertical asymptote at $x = 3$.

4. Analyze the end behavior of $f(x) = \frac{5x^3 - x + 1}{x^3 + 2x - 4}$.

Solution: Degrees equal (3 over 3), so as $x \rightarrow \pm\infty$,

$$f(x) \rightarrow \frac{5}{1} = 5.$$

Thus the horizontal asymptote is $y = 5$.

5. Analyze the end behavior of $f(x) = \frac{2x^2 - 7x + 4}{x^3 + 5x - 2}$.

Solution: Numerator degree 2, denominator degree 3, so as $x \rightarrow \pm\infty$,

$$f(x) \rightarrow 0.$$

Hence the horizontal asymptote is $y = 0$.

Function Operations & Inverses

1. Given $f(-3) = 4$, $f(0) = 2$, $f(2) = -1$ and $g(x) = 2x^2 - 1$, find $g(f(0))$.

Solution: Since $f(0) = 2$,

$$g(f(0)) = g(2) = 2 \cdot 2^2 - 1 = 8 - 1 = 7.$$

2. Given $f(0) = 3$, $f(2) = 5$, $f(4) = -1$ and $g(x) = x + 2$, find $f(g(2))$.

Solution: First compute $g(2) = 2 + 2 = 4$. Then

$$f(g(2)) = f(4) = -1.$$

3. Describe the sequence of transformations taking $f(x) = x^3$ to

$$h(x) = -2(x + 1)^3 - 4,$$

then find $h^{-1}(y)$.

Solution: Transformations:

1. Shift left 1: $(x + 1)^3$.
2. Reflect and vertical-stretch by 2: $-2(x + 1)^3$.

3. Shift down 4: $-2(x+1)^3 - 4$.

To invert, solve for x :

$$y = -2(x+1)^3 - 4 \implies y + 4 = -2(x+1)^3 \implies (x+1)^3 = -\frac{y+4}{2} \implies x = \sqrt[3]{-\frac{y+4}{2}} - 1.$$

Thus

$$h^{-1}(y) = \sqrt[3]{-\frac{y+4}{2}} - 1.$$

4. Evaluate $f^{-1}(2)$ if $f(1) = 4$, $f(2) = 2$, $f(3) = -3$, $f(4) = 1$.

Solution: We seek x such that $f(x) = 2$. From the given information, $f(2) = 2$, so

$$f^{-1}(2) = 2.$$

5. Find the inverse of $f(x) = 3x - 7$.

Solution: Write $y = 3x - 7$, swap x and y :

$$x = 3y - 7 \implies 3y = x + 7 \implies y = \frac{x+7}{3}.$$

Hence

$$f^{-1}(x) = \frac{x+7}{3}.$$

Lines, Parabolas & Circles

1. Find the equation of the line perpendicular to $y = -\frac{1}{2}x + 3$ passing through $(2, -1)$.

Solution: The slope is the negative reciprocal: $m = 2$. Using point-slope:

$$y + 1 = 2(x - 2) \implies y = 2x - 5.$$

2. Write the equation of the line parallel to $y = 7x - 2$ passing through $(-1, 3)$.

Solution: Same slope $m = 7$. Point-slope gives

$$y - 3 = 7(x + 1) \implies y = 7x + 10.$$

3. Find the vertex of the parabola $f(x) = -x^2 + 6x - 2$.

Solution: Vertex x -coordinate: $-\frac{b}{2a} = -\frac{6}{2(-1)} = 3$. Then

$$f(3) = -3^2 + 6 \cdot 3 - 2 = -9 + 18 - 2 = 7.$$

Vertex at $(3, 7)$.

4. Identify the center and radius of the circle $(x + 5)^2 + (y - 3)^2 = 36$.

Solution: In standard form $(x - h)^2 + (y - k)^2 = r^2$, we have $h = -5$, $k = 3$, $r = \sqrt{36} = 6$. So the center is $(-5, 3)$ and the radius is 6.

Optimization Problems

1. A rectangular pen is built next to a barn wall using 40 ft of fencing (two sides of length x , one side y).
(a) Express the area in terms of x and y .

Solution:

$$A = x \cdot y.$$

- (b) Write the area as a function of x alone.

Solution: From $2x + y = 40$ we get $y = 40 - 2x$. Hence

$$A(x) = x(40 - 2x) = -2x^2 + 40x.$$

- (c) Find the values of x and y that maximize the area (via the vertex formula).

Solution: The parabola $A(x) = -2x^2 + 40x$ has vertex at

$$x = -\frac{b}{2a} = -\frac{40}{2(-2)} = 10.$$

Then $y = 40 - 2 \cdot 10 = 20$, and

$$A_{\max} = 10 \cdot 20 = 200.$$

2. Find the maximum area of a rectangle with perimeter 50. Give its dimensions.

Solution: Perimeter $2x + 2y = 50 \rightarrow y = 25 - x$. So

$$A(x) = x(25 - x) = -x^2 + 25x,$$

vertex at

$$x = -\frac{25}{2(-1)} = 12.5, \quad y = 25 - 12.5 = 12.5.$$

Thus the maximum occurs at a 12.5×12.5 square, and $A_{\max} = 156.25$.

3. A revenue model is $R(p) = -5p^2 + 100p$.

(a) Find the price p that maximizes revenue.

Solution: The parabola $R(p) = -5p^2 + 100p$ has vertex at

$$p = -\frac{100}{2(-5)} = 10.$$

(b) Compute the maximum revenue.

Solution:

$$R(10) = -5(10)^2 + 100 \cdot 10 = -500 + 1000 = 500.$$

Logarithmic & Exponential Functions

1. Evaluate $\log_4\left(\frac{1}{64}\right)$.

Solution: Since $64 = 4^3$, $\frac{1}{64} = 4^{-3}$. Thus

$$\log_4\left(\frac{1}{64}\right) = -3.$$

2. Simplify $\log_2(16) + \log_2\left(\frac{1}{2}\right)$.

Solution: $\log_2(16) = 4$ and $\log_2\left(\frac{1}{2}\right) = -1$, so the sum is

$$4 + (-1) = 3.$$

3. Convert $\ln(y) = 7$ into exponential form.

Solution: Exponentiating both sides with base e :

$$y = e^7.$$

4. Convert $\ln(8) = y$ into exponential form.

Solution: Exponential form: $e^y = 8$. Taking the natural log again confirms

$$y = \ln(8).$$

5. Solve $\log_3(x^2 + 1) = 2$.

Solution: Rewrite in exponential form:

$$x^2 + 1 = 3^2 = 9 \implies x^2 = 8 \implies x = \pm 2\sqrt{2}.$$

6. Solve $\log_2(x^2 - 4) = 3$. State any restrictions.

Solution: Exponential form:

$$x^2 - 4 = 2^3 = 8 \implies x^2 = 12 \implies x = \pm 2\sqrt{3}.$$

Restriction: $x^2 - 4 > 0$, so $x < -2$ or $x > 2$. Both solutions satisfy this.

7. Evaluate $\log_3(9) - \log_3(\frac{1}{3})$.

Solution: $\log_3(9) = 2$ and $\log_3(\frac{1}{3}) = -1$, so

$$2 - (-1) = 3.$$

Trigonometric Evaluation & Identities

1. Evaluate $\tan(\frac{5\pi}{6})$.

Solution: $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, so

$$\tan(\frac{5\pi}{6}) = -\tan(\frac{\pi}{6}) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

2. Simplify $\frac{\sin^2(x) + \cos^2(x)}{\tan(x)}$.

Solution: Use $\sin^2 x + \cos^2 x = 1$ and $\tan x = \frac{\sin x}{\cos x}$:

$$\frac{1}{\tan x} = \cot x.$$

3. Simplify $\frac{1 - \cos(2x)}{\sin(x)}$.

Solution: Use $1 - \cos(2x) = 2\sin^2(x)$:

$$\frac{2\sin^2(x)}{\sin(x)} = 2\sin(x).$$

4. Evaluate $\sin\left(\frac{7\pi}{3}\right)$ by finding a coterminal angle.

Solution: $\frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$. Thus $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

5. Use a sum identity to evaluate $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$.

Solution:

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}.$$

6. Use a sum identity to compute $\sin\left(\frac{5\pi}{12}\right)$.

Solution: Write $\frac{5\pi}{12} = \frac{\pi}{3} + \frac{\pi}{12}$. Then

$$\sin\left(\frac{5\pi}{12}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{12} + \cos\frac{\pi}{3}\sin\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

7. Use a half-angle identity to evaluate $\sin\left(\frac{5\pi}{8}\right)$.

Solution: Since $\frac{5\pi}{8} = \frac{1}{2}\left(\frac{5\pi}{4}\right)$ and $\sin$ is positive:

$$\sin\left(\frac{5\pi}{8}\right) = \sqrt{\frac{1 - \cos\left(\frac{5\pi}{4}\right)}{2}} = \sqrt{\frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

8. Given $\cos(\theta) = \frac{4}{5}$ with $0 < \theta < \frac{\pi}{2}$, find $\cos(2\theta)$.

Solution: Use $2\cos^2\theta - 1$:

$$2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}.$$

9. Given $\sin(\phi) = \frac{7}{25}$ with $\frac{\pi}{2} < \phi < \pi$, find $\cos\left(\frac{\phi}{2}\right)$.

Solution: Here $\phi/2 \in (\frac{\pi}{4}, \frac{\pi}{2})$, so cosine is positive. First $\cos \phi = -\sqrt{1 - \sin^2 \phi} = -\frac{24}{25}$. Then

$$\cos\left(\frac{\phi}{2}\right) = \sqrt{\frac{1 + \cos \phi}{2}} = \sqrt{\frac{1 - \frac{24}{25}}{2}} = \frac{\sqrt{2}}{10}.$$

10. Given $\cos(\theta) = \frac{5}{13}$ with $0 < \theta < \frac{\pi}{2}$, find $\cos(2\theta)$.

Solution:

$$\cos(2\theta) = 2\cos^2 \theta - 1 = 2\left(\frac{5}{13}\right)^2 - 1 = \frac{50}{169} - 1 = -\frac{119}{169}.$$

Inverse Trigonometric Functions

1. Evaluate $\arcsin(-1)$.

Solution: By definition, $\arcsin(-1) = -\frac{\pi}{2}$.

2. Evaluate $\arccos(-\frac{1}{2})$.

Solution: $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$.

3. Evaluate $\tan^{-1}(1)$ and explain the result.

Solution: Since $\tan(\frac{\pi}{4}) = 1$ and the range of $\tan^{-1}$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan^{-1}(1) = \frac{\pi}{4}$. Hence $\pi/4$ is the angle whose slope (tangent) is 1.

4. Evaluate $\cos^{-1}(\cos(\frac{11\pi}{6}))$.

Solution: $\cos(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2}$. Then $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$.

Trigonometric Equations

1. Find all solutions to $2\cos(3x) = 1$ for $0 \leq x \leq 2\pi$.

Solution:

$$\cos(3x) = \frac{1}{2} \implies 3x = 2\pi k \pm \frac{\pi}{3} \implies x = \frac{2\pi k}{3} \pm \frac{\pi}{9}.$$

For $k = 0, 1, 2$ this yields

$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}.$$

2. Find all solutions to $\cos^2(x) - \sin(x) = 1$ in $[0, 4\pi]$.

Solution: Rewrite:

$$1 - \sin^2(x) - \sin(x) = 1 \implies -\sin^2(x) - \sin(x) = 0 \implies \sin(x)(\sin(x) + 1) = 0.$$

Hence $\sin(x) = 0 \rightarrow x = n\pi$, $n = 0, 1, 2, 3, 4$, and $\sin(x) = -1 \rightarrow x = \frac{3\pi}{2}, \frac{7\pi}{2}$.

3. Find all solutions to $2\sin(x) = \sqrt{3}$ in $[0, 2\pi]$.

Solution: $\sin(x) = \frac{\sqrt{3}}{2} \rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$.

4. Solve $\sin^2(x) + \sin(x) = 0$ for all real x .

Solution: Factor: $\sin(x)(\sin(x) + 1) = 0$, so $\sin(x) = 0 \rightarrow x = n\pi$, or $\sin(x) = -1 \rightarrow x = \frac{3\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$.

Triangle Applications

1. A 12-ft ladder leans against a wall, making a 60° angle with the ground. How high up the wall does it reach?

Solution: Height $= 12 \sin(60^\circ) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$ ft.

2. In a triangle with angles 30° , 60° and hypotenuse 8, find the side opposite 30° .

Solution: Opposite $30^\circ = 8 \sin(30^\circ) = 8 \cdot \frac{1}{2} = 4$.

3. In a right triangle with hypotenuse 10 and one angle 30° , find the side opposite 30° .

Solution: Opposite $30^\circ = 10 \sin(30^\circ) = 10 \cdot \frac{1}{2} = 5$.

4. In triangle ABC, $a = 6$, $b = 7$, $\angle C = 45^\circ$. Find c via the Law of Cosines.

Solution:

$$c^2 = a^2 + b^2 - 2ab \cos C = 36 + 49 - 2 \cdot 6 \cdot 7 \cdot \frac{\sqrt{2}}{2} = 85 - 42\sqrt{2},$$

so $c = \sqrt{85 - 42\sqrt{2}}$.

5. In a triangle with $a = 5$, $b = 7$, $\angle C = 60^\circ$, use the Law of Cosines to find c .

Solution:

$$c^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos 60^\circ = 25 + 49 - 35 = 39,$$

hence $c = \sqrt{39}$.

Graphing & Transformations

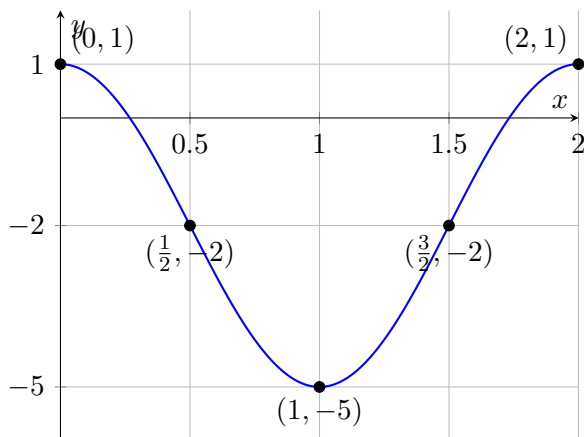
1. State the amplitude, period, and midline of $f(x) = 3 - 4\cos(2x)$.

Solution: The form is $A \cos(Bx) + D$. Here $A = -4$ so amplitude $= |A| = 4$. $B = 2$ so period $= \frac{2\pi}{B} = \pi$. Midline $= D = 3$.

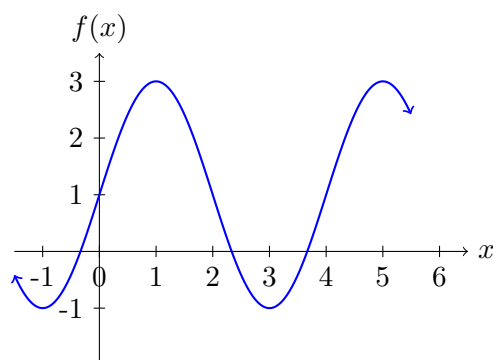
2. Sketch one period of $y = 3\cos(\pi x) - 2$, labeling key points.

Solution:

- Amplitude $= 3$, midline $y = -2$, period $= \frac{2\pi}{\pi} = 2$.
- Key x -values: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$.
- At $x = 0$: $y = 3(-2) = 1$.
- At $x = \frac{1}{2}$: $\cos(\frac{\pi}{2}) = 0 \rightarrow y = -2$.
- At $x = 1$: $\cos(\pi) = -1 \rightarrow y = -5$.
- At $x = \frac{3}{2}$: $\cos(\frac{3\pi}{2}) = 0 \rightarrow y = -2$.
- At $x = 2$: $\cos(2\pi) = 1 \rightarrow y = 1$.



3. What is the equation of the function whose graph is shown below?



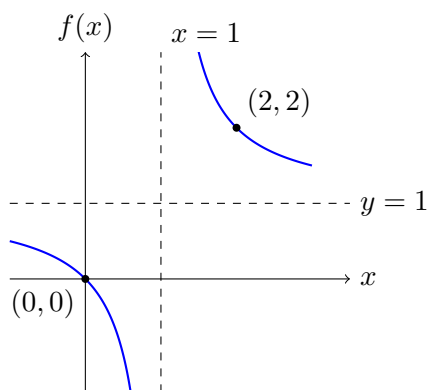
Solution: The graph is $1 + 2\sin\left(\frac{\pi}{2}x\right)$. Amplitude 2, midline $y = 1$, period $= \frac{2\pi}{\pi/2} = 4$.

4. Sketch a function f satisfying:

- $\lim_{x \rightarrow 1^-} f(x) = -\infty$
- $\lim_{x \rightarrow 1^+} f(x) = \infty$
- $f(0) = 0, f(2) = 2$
- Horizontal asymptote $y = 1$

Solution: Draw a curve with:

- A vertical asymptote at $x = 1$, rising to $+\infty$ from the left, falling to $-\infty$ from the right.
- It passes through $(0, 0)$ and $(2, 2)$.
- Levels off toward $y = 1$ as $x \rightarrow \pm\infty$.



5. Sketch the graph of a function $g(x)$ satisfying all of the following properties:

- As $x \rightarrow 2^-$, $g(x) \rightarrow 5$.
- As $x \rightarrow 2^+$, $g(x) \rightarrow -3$.
- $g(2) = 3$
- $g(0) = 1$.

- $\lim_{x \rightarrow -1} g(x) = 2$ but $g(-1) = 0$
- As $x \rightarrow +\infty$, $g(x) \rightarrow 0$
- As $x \rightarrow -\infty$, $g(x) \rightarrow 1$

Solution: One possible sketch is shown below.

