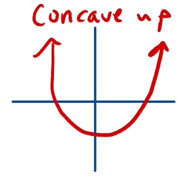
Quadratic Functions

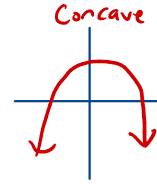
Introduction

Quadratic functions are fundamental mathematical models used in a variety of real-world contexts, such as physics, economics, and engineering. In this lecture, we will explore their properties, forms, and graphical representations.

Overview of Quadratic Function Forms

Form	Equation	Notes
General Form	$f(x) = ax^2 + bx + c$	a, b, and c are constants. Useful for identifying the y-intercept, c.
Vertex (Standard) Form	$f(x) = a(x-h)^2 + k$	a determines the direction and width of the parabola. (h, k) is the vertex.
Factored Form	$f(x) = a(x-r_1)(x-r_2)$	r_1 and r_2 are the roots (zeros) of the quadratic function. Useful for solving equations.





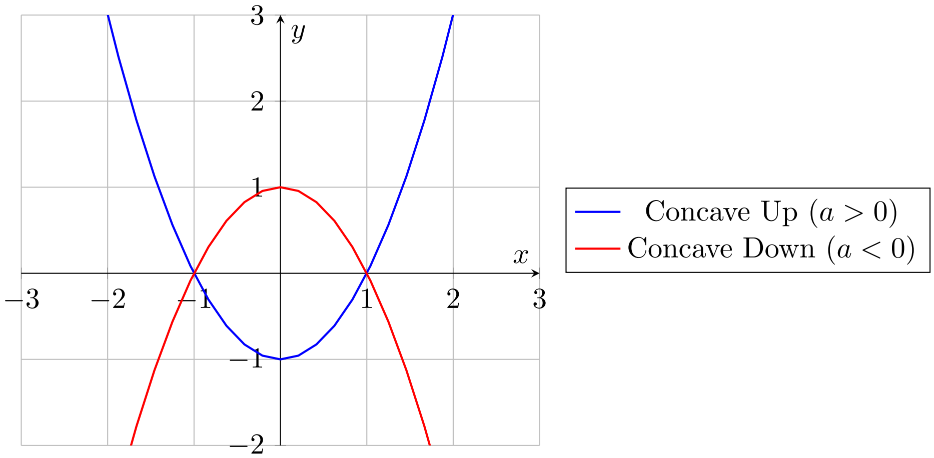
Remark. In general form,

- a determines the concavity of the function.
- b affects the symmetry across the y-axis.
- c is the y-intercept.

Concavity

The concavity of a quadratic function describes the direction the parabola opens.

- $a > 0$: parabola opens upwards (concave up).
- $a < 0$: parabola opens downwards (concave down).



Vertex and Maximum/Minimum

The vertex represents the highest or lowest point on the parabola.

- If $a > 0$, the vertex is a minimum.
- If $a < 0$, the vertex is a maximum.

Question. In standard form $y = a(x-h)^2 + k$, the vertex is (h, k). How can we find the vertex if the function is in general form $y = ax^2 + bx + c$?

In general form, the vertex occurs at $x = \frac{-b}{2a}$

The vertex is at $\left(\underset{\text{x-coord}}{\frac{-b}{2a}}, \underset{\text{y-coord}}{f\left(\frac{-b}{2a}\right)} \right)$

Finding the Vertex

Example. Find the vertex of $y = 2x^2 - 4x + 1$.

① use $x = \frac{-b}{2a}$. $x = \frac{-(-4)}{2 \cdot 2} = \frac{4}{4} = 1$

② Find y-coord. $y = 2(1)^2 - 4(1) + 1 = 2 - 4 + 1 = -1$

Answer $\boxed{(1, -1)}$

Converting General Form to Standard Form

Example. Convert $y = 2x^2 - 4x + 1$ to standard form.

From the above, $(h, k) = (1, -1)$ and $a = 2$

$$y = a(x-h)^2 + k = 2(x-1)^2 - 1$$

Finding Intercepts

Example. Find the x-intercepts and y-intercept of the quadratic function: $f(x) = 2x^2 - 4x + 1$.

① y-intercept : $f(0) = 2 \cdot 0^2 - 4 \cdot 0 + 1 = 0 - 0 + 1 = 1$

② x-intercepts : we need to solve $f(x) = 0$

$2x^2 - 4x + 1 = 0$ ↙ Not so clear

How to solve $ax^2 + bx + c = 0$

Can't factor $2x^2 - 4x + 1$ { 1. Factor

Next week { 2. Complete the square
3. Quadratic Formula

Example. Find a function f whose graph is a parabola with the given vertex and that passes through the given point.

- Vertex: $(2, -3)$
- Point: $(4, 5)$

Vertex / Standard Form: $f(x) = a(x-h)^2 + k$

Substitute $(2,-3)$: $f(x) = a(x-2)^2 - 3$

Substitute $(4,5)$ to find a: $f(4) = a(4-2)^2 - 3 = 5$

$$a(2)^2 - 3 = 5$$

$$4a = 8$$

$$a = 2$$

$\Rightarrow$ $\boxed{f(x) = 2(x-2)^2 - 3}$

Graphing Quadratic Functions

Example. Graph the quadratic function $f(x) = x^2 - 2x - 3$ and determine its domain and range.

① Vertex: $x = \frac{-b}{2a} = \frac{-(-2)}{2 \cdot 1} = \frac{2}{2} = 1$

$y = (1)^2 - 2(1) - 3 = 1 - 2 - 3 = -4$

The vertex is $\boxed{(1,-4)}$

② y-intercept is $f(0) = 0^2 - 2 \cdot 0 - 3 = -3$ $\boxed{(0,-3)}$

③ x-intercepts: Solve $f(x) = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

$\boxed{(3,0)}$ and $\boxed{(-1,0)}$

④ Choose additional points on the graph:

$f(2) = 2^2 - 2(2) - 3 = -3$ $\boxed{(2,-3)}$

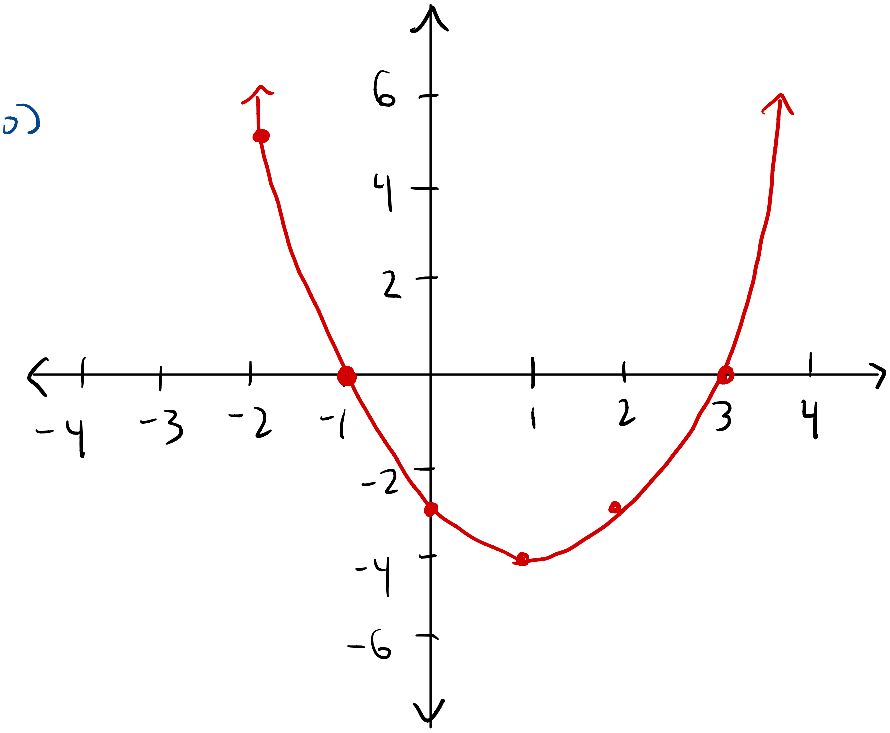
$f(-2) = (-2)^2 - 2(-2) - 3 = 5$ $\boxed{(-2,5)}$

Vertex: $(1,-4)$

y-intercept: $(0,-3)$

x-intercepts: $(3,0)$ and $(-1,0)$

points: $(2,-3)$ and $(-2,5)$



Domain: $(-\infty,\infty)$

Range: $[-4,\infty)$

Application: Projectile Motion

In physics, the motion of an object under the influence of gravity, such as a ball thrown in the air, can be modeled by a quadratic function. The equation for the height $h(t)$ of the object at time t is typically given by:

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0,$$

where:

- g is the acceleration due to gravity (approximately 9.8 m/s^2 on Earth).
- v_0 is the initial velocity of the object (in m/s).
- h_0 is the initial height of the object (in m).

Example. A ball is thrown upward with an initial velocity of 20 m/s from a height of 2 m. The height of the ball at any time t seconds is given by:

$$h(t) = -4.9t^2 + 20t + 2.$$

Find:

1. The maximum height of the ball.
2. The time it takes for the ball to hit the ground.