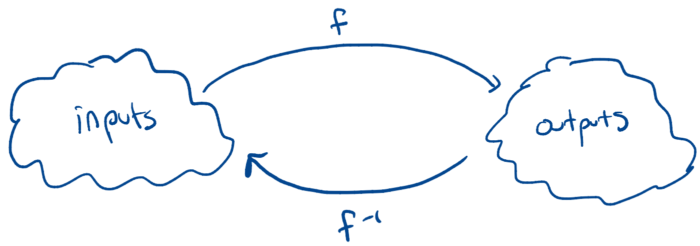
Inverse Functions

Inverse functions allow us to reverse the process of a function, turning the output (y) back into the input (x). We will explore what makes a function invertible, how to find inverses, and their connection to reflections about the line $y = x$.

Definition. What is an inverse function?

If a function f sends x to y, then the inverse function sends y to x



If $f(x) = y$

then $f^{-1}(y) = x$

Question. How can we verify if two functions are inverses of each other?

We can use the Inverse Function Property. This says that if we compose the two functions, we get back what we started with.

In particular,

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

Example. If $f(x) = 2x + 3$, verify that $f^{-1}(x) = \dfrac{x-3}{2}$.

Example: Try $x = 2$. $f(2) = 2 \cdot 2 + 3 = 7$

$$f^{-1}(7) = \frac{7-3}{2} = \frac{4}{2} = 2 \checkmark$$

$$f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x$$

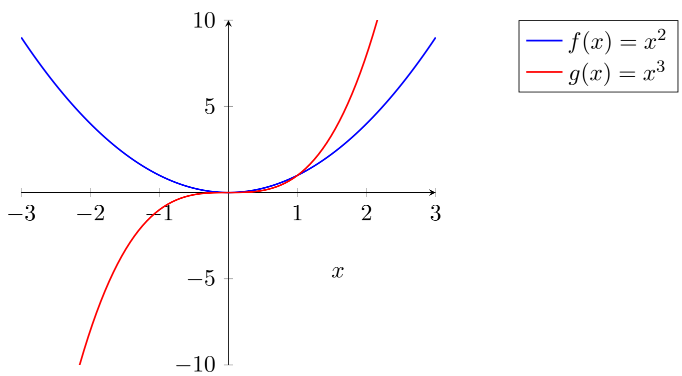
$$f(f^{-1}(y)) = f\left(\frac{y-3}{2}\right) = 2 \cdot \left(\frac{y-3}{2}\right) + 3 = y - 3 + 3 = y$$

Question. When is a function invertible?

A function has an inverse if it is one-to-one, meaning each y-value (output) corresponds to exactly one x-value (input)

Horizontal Line Test: A function is one-to-one if any horizontal line intersects its graph at most once.

Example. Determine whether or not x^2 and x^3 are invertible.



$f(x) = x^2$ is NOT invertible because it fails the horizontal line test.

$f^{-1}(4)$ has TWO possibilities. 2 and -2

$g(x) = x^3$ is invertible because it passes the horizontal line test.

Question. Given a function $f(x)$, what are the steps to find the inverse function $f^{-1}(x)$?

1. Denote the output of $f(x)$ by the variable y
2. By the Inverse Function Property, $f^{-1}(y) = x$, so solve for x in terms of y
3. This gives a formula for $f^{-1}(y)$.

Example. Find the inverse of $f(x) = \dfrac{2x+1}{x-3}$.

Let $y = \dfrac{2x+1}{x-3}$

I need to solve for x in terms of y

$$y(x-3) = 2x+1$$

$$yx - 3y = 2x+1$$

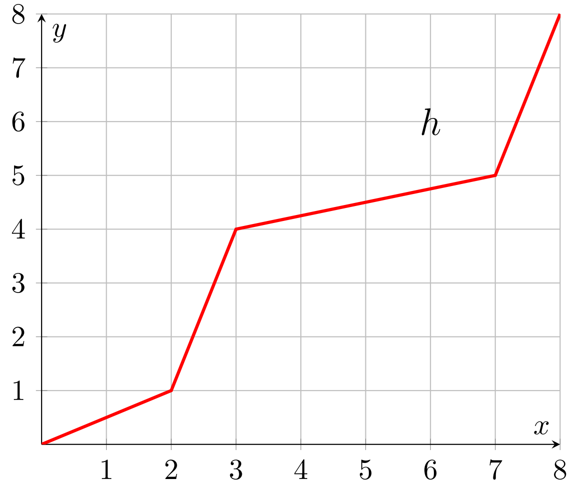
$$yx - 2x = 3y+1$$

$$x(y-2) = 3y+1$$

$$x = \frac{3y+1}{y-2}$$

Hence $f^{-1}(y) = \dfrac{3y+1}{y-2}$. If you prefer to call the inputs to your functions x, you can write $f^{-1}(x) = \dfrac{3x+1}{x-2}$

Example. A graph of a function is given. Use the graph to find the indicated values.



(a) $h^{-1}(2) =$ 2.4

(b) $h^{-1}(3) =$ 2.8

(c) $h^{-1}(7) =$ 7.6

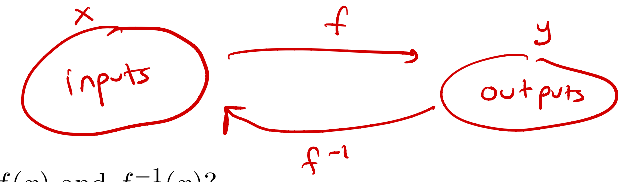
Example. If $f(x) = 3x + 1$, what is $f^{-1}(10)$?

We need to solve $3x + 1 = 10$

$3x = 9$

$x = 3$

Hence $f(3) = 10$ and $f^{-1}(10) = 3$

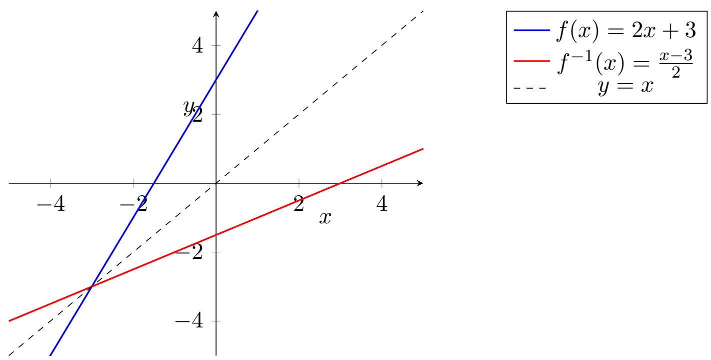


Question. What is the relationship between the graphs of $f(x)$ and $f^{-1}(x)$?

The graph of f^{-1} is a reflection of f about the line $y=x$.

Note: If (a,b) is on the graph of $f(x)$, then (b,a) will be on the graph of $f^{-1}(x)$

Example. Below is the graph of $f(x) = 2x + 3$ and its inverse function $f^{-1}(x) = \dfrac{x-3}{2}$.



Question. What is the range of $f(x)$?

The range of $f(x)$ is the domain of $f^{-1}(x)$.

In particular, the outputs of $f(x)$ are precisely the inputs of $f^{-1}(x)$.

Example. What is the range of $f(x) = \dfrac{2x+1}{x-3}$?

We saw that $f^{-1}(x) = \dfrac{3x+1}{x-2}$

The domain of $f^{-1}(x)$ is $x \neq 2$

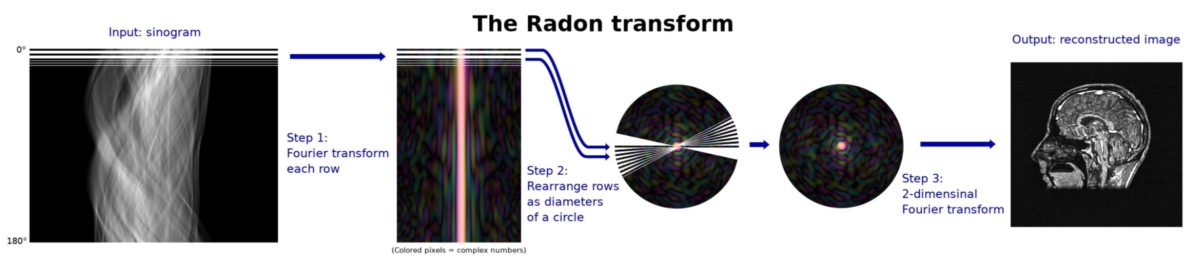
Conclude: The range of $f(x)$ is $y \neq 2$

Application: CT scans

Medical imaging techniques, like CT (Computed Tomography) scans, use inverse functions to reconstruct images of the inside of the human body from external measurements.

How It Works

1. **Data Collection**:
   - A CT scanner rotates around the patient, taking multiple X-ray images from different angles.
   - These X-ray images measure how much energy passes through the body at each angle, creating a dataset of projections.
2. **The Problem**:
   - The projections (raw data) don't directly show the internal structure of the body.
   - To create an image of the body, we need to work backward from the projections to find the original density distribution of tissues inside the body.



3. **Using Inverse Functions**:
   - The process of reconstructing the image involves solving a mathematical problem called the **Radon Transform**.
   - In simple terms, the CT scanner applies an inverse function to the collected data to "undo" the projection process, reconstructing the original image.
4. **The Result**:
   - The reconstructed image shows cross-sections of the body, helping doctors identify abnormalities like tumors, fractures, or internal bleeding.

Why Inverse Functions Matter in CT Scans

- Without inverse functions, the raw data collected by the scanner would be meaningless, as it only shows how much energy passed through the body—not the actual structure.
- Inverse functions help "translate" the data into a detailed image, allowing doctors to make accurate diagnoses.