Transformations of Functions

Introduction

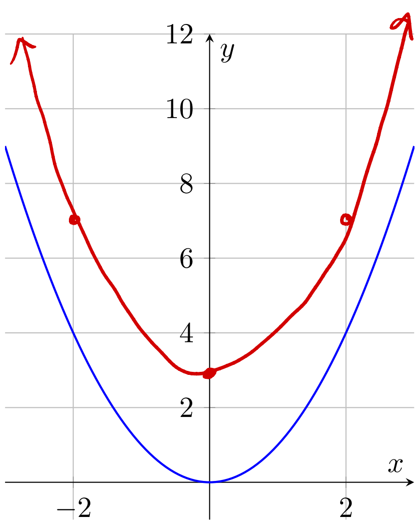
Transformations allow us to modify functions to shift, stretch, compress, or reflect their graphs. Understanding transformations is key to graphing functions quickly and interpreting their behavior.

Types of Transformations

1. Vertical Transformations:

Transformation	Formula	What to do to the Coordinates
Vertical shift (up c)	$f(x)+c$	Add c to each y-coordinate $y \mapsto y+c$
Vertical shift (down c)	$f(x)-c$	Subtract c from each y-coordinate $y \mapsto y-c$
Vertical stretch	$a \cdot f(x),\ a>1$	Multiply each y-coordinate by a $y \mapsto a \cdot y$
Vertical shrink	$a \cdot f(x),\ 0<a<1$	Multiply each y-coordinate by a $y \mapsto a \cdot y$
Reflection about the x-axis	$-f(x)$	Multiply each y-coordinate by -1 $y \mapsto -y$

Example. Consider the quadratic function $f(x) = x^2$. A **vertical shift up by 3** transforms the function to $f(x)+3 = x^2+3$.



x	$f(x)=x^2$	$f(x)+3$
−3	9	12
−2	4	7
−1	1	4
0	0	3
1	1	4
2	4	7
3	9	12

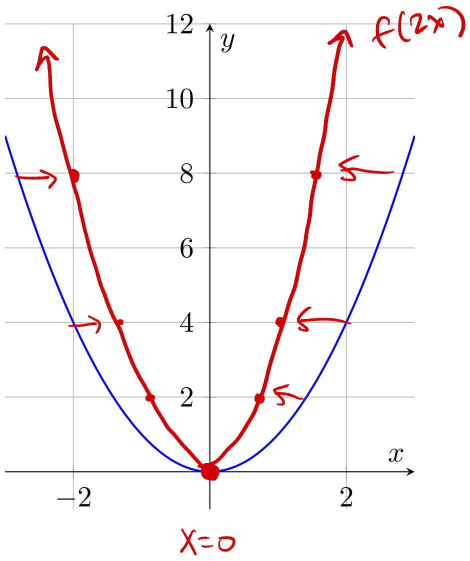
* Note: vertical transformations only affect the y-coordinate.

* These change the input of the function (i.e. the x-coords)

2. Horizontal Transformations:

Transformation	Formula	What to do to the Coordinates
Horizontal shift (right c)	$f(x-c)$	Add c to each x-coord. $x \mapsto x+c$
Horizontal shift (left c)	$f(x+c)$	Subtract c from each x-coord $x \mapsto x-c$
Horizontal stretch	$f(b\cdot x)$, $0<b<1$	Multiply each x-coord by $1/b$ $x \mapsto x/b$
Horizontal shrink	$f(bx)$, $b>1$	Multiply each x-coord by $1/b$ $x \mapsto x/b$
Reflection about the y-axis	$f(-x)$	Multiply each x-coord by -1 $x \mapsto -x$

Example. Consider the quadratic function $f(x) = x^2$. A **horizontal shrink by a factor of 2** transforms the function to $f(2x)$.



x	$f(x) = x^2$	$f(2x)$
-3	9	36
-2	4	16
-1	1	4
0	0	0
1	1	4
2	4	16
3	9	36

Sequences of Transformations

Question. How do we apply more than one transformation at a time?

Transformations are applied to the resulting function from the previous step

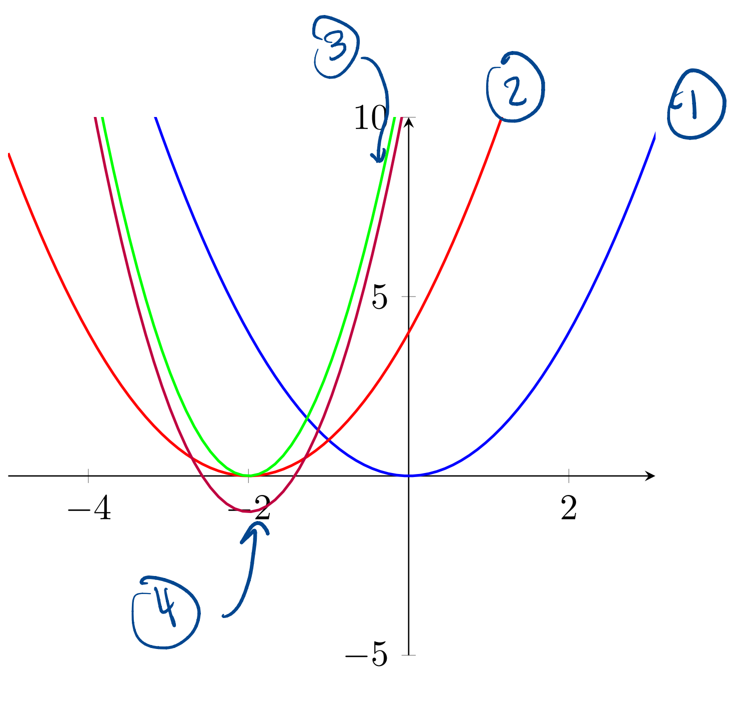
Example: Let $f(x) = x^2$. Apply the following transformations in order:

1. Horizontal shift left 2.
2. Vertical stretch by 3.
3. Vertical shift down 1.

1. $f(x+2) = (x+2)^2$

2. $3 \cdot f(x+2) = 3 \cdot (x+2)^2$

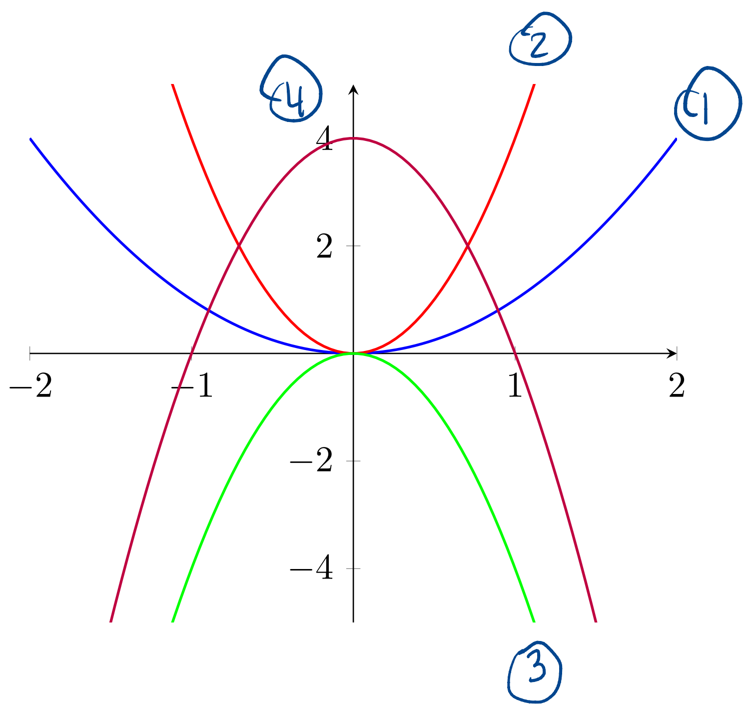
3. $3 \cdot f(x+2) - 1 = 3 \cdot (x+2)^2 - 1$



Example. Let $f(x) = x^2$. Apply the following transformations in order:

1. Horizontal shrink by a factor of 2.
2. Reflection about the x-axis.
3. Vertical shift up 4.

As a transformation of $f(x)$	As an explicit equation
$f(2x)$	$(2x)^2$
$-f(2x)$	$-(2x)^2$
$-f(2x)+4$	$-(2x)^2+4$



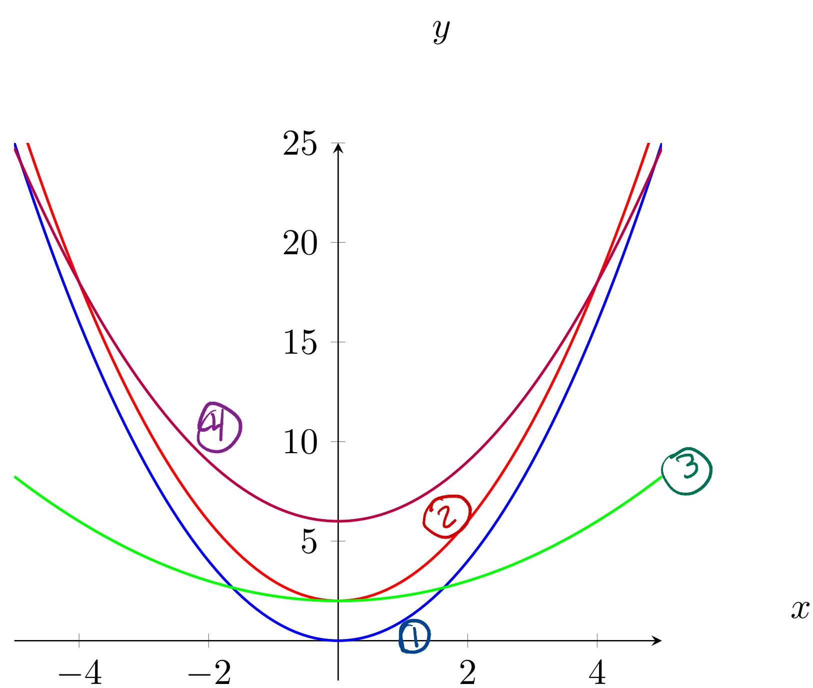
Example. Let $f(x) = x^2$. Apply the following transformations in order:

1. Vertical shift up by 2.
2. Horizontal stretch by a factor of 2.
3. Vertical stretch by a factor of 3.

What is a simplified formula for the resulting transformation?

As a transformation of f(x)	As an explicit formula
$f(x)+2$	x^2+2
$f(\frac{x}{2})+2$	$(\frac{x}{2})^2+2$
$3(f(\frac{x}{2})+2)$	$3((\frac{x}{2})^2+2)$

$$\boxed{3\left(\frac{x}{2}\right)^2+6}$$



Example. Given the transformed function:

$$g(x) = -\sqrt{x-3} + 2$$

Identify a parent function $f(x)$ and state the transformations, in order, needed to get from $f(x)$ to $g(x)$.

1 Here, we can let $f(x) = \sqrt{x}$

2 Write $g(x)$ as a transformation of $f(x)$

$$g(x) = -f(x-3) + 2$$

3 1. Horizontal shift right by 3 $\quad f(x) \longmapsto f(x-3)$

2. Reflect about x-axis $\quad f(x-3) \longmapsto -f(x-3)$

3. Vertical shift up 2 $\quad -f(x-3) \longmapsto -f(x-3)+2$

Example. Given the transformed function:

$$h(x) = 2(x+1)^2 - 4$$

Identify a parent function $f(x)$ and state the transformations, in order, needed to get from $f(x)$ to $h(x)$.

1. Parent function is $f(x) = x^2$

2. $h(x) = 2f(x+1) - 4$

3. Transformations:

   1. horizontal shift left 1 $\quad f(x+1)$

   2. Vertical stretch by factor of 2 $\quad 2f(x+1)$

   3. shift down 4 $\quad 2f(x+1)-4$