Composition of Functions

Introduction

Functions can be combined in many ways to create new functions, including addition, subtraction, multiplication, division, and composition.

Combining Functions: Addition, Subtraction, Multiplication, and Division

Combinations of Functions

Let $f(x)$ and $g(x)$ be two functions. Their combinations are defined as:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\\ (f-g)(x) &= f(x)-g(x),\\ (f\cdot g)(x) &= f(x)\cdot g(x),\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \quad g(x)\neq 0.\end{aligned}$$

Example. Let $f(x) = x^2+1$ and $g(x) = 3x-4$. Find $(f+g)(x), (f-g)(x), (f\cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$.

Example. Let $f(x) = \sqrt{x}$ and $g(x) = x-1$. Find $(f\cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Composing of Functions

Definition. The composition of f and g, written $(f \circ g)(x)$, means $f(g(x))$. The output of $g(x)$ becomes the input for $f(x)$.

Steps to Compute $(f \circ g)(x)$

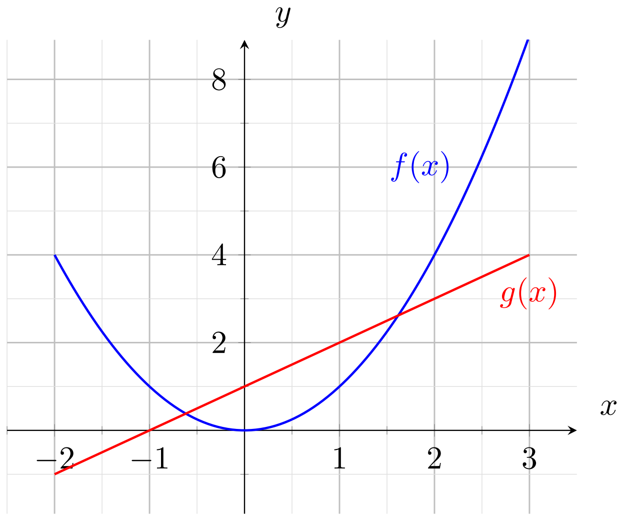
1. Substitute $g(x)$ into $f(x)$.
2. Simplify the resulting expression.

Example. Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

Example. Let $f(x) = \sqrt{x + 1}$ and $g(x) = x^2$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

Evaluating Combined Functions From a Graph

Given the graphs of $f(x)$ and $g(x)$, evaluate the following combined functions.



Example. Evaluate $(f+g)(2)$

Example. Evaluate $(f-g)(-1)$.

Example. Evaluate $(f \circ g)(1)$.

Example. Evaluate $(f \cdot g)(0)$.

Example. Evaluate $\left(\frac{f}{g}\right)(-2)$.

Evaluating Combined Functions From a Table

x	$f(x)$	$g(x)$
-1	1	0
0	0	1
1	1	2

Example. Evaluate $(f+g)(1)$

Example. Evaluate $(f \cdot g)(-1)$

Example. Evaluate $(f \circ g)(0)$

Domain of Combined Functions

The domain of a combined function depends on the domains of $f(x)$ and $g(x)$, as well as the operation being performed:

General Rules for Domains

- **Addition/Subtraction:** For $(f+g)(x)$ or $(f-g)(x)$, the domain is the intersection of the domain of f and the domain of g.
- **Multiplication:** For $(f \cdot g)(x)$, the domain is the intersection of the domain of f and the domain of g.
- **Division:** For $\left(\frac{f}{g}\right)(x)$, the domain is the intersection of the domain of f and the domain of g, and we also exclude values where $g(x) = 0$.
- **Composition:** For $(f \circ g)(x)$, x must belong to the domain of g, and $g(x)$ must belong to the domain of f.

Example. Let $f(x) = \sqrt{x}$ and $g(x) = x - 2$. Find the domain of $(f \circ g)(x)$.

Example. Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 4$. Find the domain of $\left(\frac{f}{g}\right)(x)$.

Application: Calories Burned as a Function of Time

Fitness tracking often involves calculating the number of calories burned based on the time spent exercising. Suppose you know two things:

1. **Distance as a Function of Time:**

$$d(t) = 6t$$

   - Where t is the time in hours, and $d(t)$ is the distance (in miles) walked in that time.

2. **Calories Burned as a Function of Distance:**

$$C(d) = 100d$$

   - Where d is the distance (in miles), and $C(d)$ is the total calories burned.

We can use function **composition** to calculate the total calories burned as a function of time:

$$C(d(t)) = C(6t) = 100(6t) = 600t.$$

This composition directly links walking time to calories burned. For every hour of walking, the person burns 600 calories.