Composition of Functions

Introduction

Functions can be combined in many ways to create new functions, including addition, subtraction, multiplication, division, and composition.

Combining Functions: Addition, Subtraction, Multiplication, and Division

Combinations of Functions

Let $f(x)$ and $g(x)$ be two functions. Their combinations are defined as:

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x),$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$$

Example. Let $f(x) = x^2 + 1$ and $g(x) = 3x - 4$. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$.

$$(f+g)(x) = f(x) + g(x) = (x^2+1) + (3x-4) = x^2 + 3x - 3$$

$$(f-g)(x) = f(x) - g(x) = (x^2+1) - (3x-4) = x^2 - 3x + 5$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = (x^2+1) \cdot (3x-4) = 3x^3 - 4x^2 + 3x - 4$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+1}{3x-4}$$

Example. Let $f(x) = \sqrt{x}$ and $g(x) = x - 1$. Find $(f \cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$.

$$(f \cdot g)(x) = \sqrt{x}\,(x-1)$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x-1}$$

Composing of Functions

Definition. The composition of f and g, written $(f \circ g)(x)$, means $f(g(x))$. The output of $g(x)$ becomes the input for $f(x)$.

Steps to Compute $(f \circ g)(x)$
1. Substitute $g(x)$ into $f(x)$.   2. Simplify the resulting expression.

Example. Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1)$$
$$= 2(x^2 - 1) + 3$$
$$= 2x^2 - 2 + 3$$
$$= 2x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(2x + 3)$$
$$= (2x+3)^2 - 1$$
$$= 4x^2 + 12x + 9 - 1$$
$$= 4x^2 + 12x + 8$$

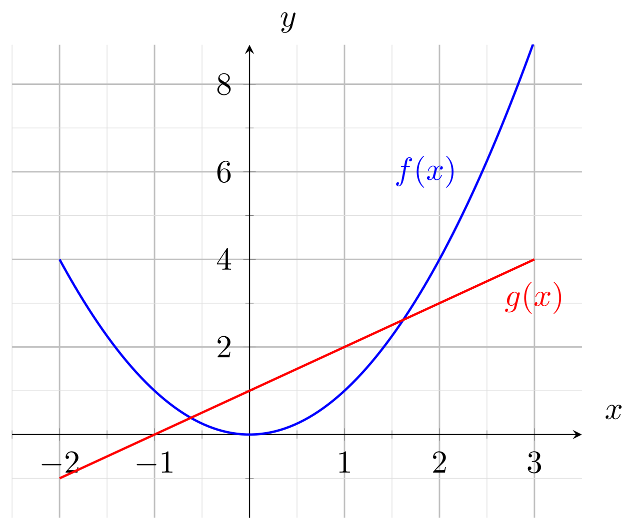
Example. Let $f(x) = \sqrt{x+1}$ and $g(x) = x^2$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{x^2+1}$$

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = (\sqrt{x+1})^2 = x + 1$$

Evaluating Combined Functions From a Graph

Given the graphs of $f(x)$ and $g(x)$, evaluate the following combined functions.



Example. Evaluate $(f+g)(2)$

$$(f+g)(2) = f(2)+g(2) = 4+3 = 7$$

Example. Evaluate $(f-g)(-1)$.

$$(f-g)(-1) = f(-1)-g(-1) = 1-0 = 1$$

Example. Evaluate $(f\circ g)(1)$.

$$(f\circ g)(1) = f(g(1)) = f(2) = 4$$

Example. Evaluate $(f\cdot g)(0)$.

$$(f\cdot g)(0) = f(0)\cdot g(0) = 0\cdot 1 = 0$$

Example. Evaluate $\left(\frac{f}{g}\right)(-2)$.

$$\left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)} = \frac{4}{-1} = -4$$

Evaluating Combined Functions From a Table

x	$f(x)$	$g(x)$
-1	1	0
0	0	1
1	1	2

Example. Evaluate $(f+g)(1)$

$$(f+g)(1) = f(1)+g(1) = 1+2 = 3$$

Example. Evaluate $(f \cdot g)(-1)$

$$(f \cdot g)(-1) = f(-1) \cdot g(-1) = 1 \cdot 0 = 0$$

Example. Evaluate $(f \circ g)(0)$

$$(f \circ g)(0) = f(g(0)) = f(1) = 1$$

Domain of Combined Functions

The domain of a combined function depends on the domains of $f(x)$ and $g(x)$, as well as the operation being performed:

General Rules for Domains

- **Addition/Subtraction:** For $(f+g)(x)$ or $(f-g)(x)$, the domain is the intersection of the domain of f and the domain of g.
- **Multiplication:** For $(f \cdot g)(x)$, the domain is the intersection of the domain of f and the domain of g.
- **Division:** For $\left(\frac{f}{g}\right)(x)$, the domain is the intersection of the domain of f and the domain of g, and we also exclude values where $g(x) = 0$.
- **Composition:** For $(f \circ g)(x)$, x must belong to the domain of g, and $g(x)$ must belong to the domain of f.

Example. Let $f(x) = \sqrt{x}$ and $g(x) = x - 2$. Find the domain of $(f \circ g)(x)$.

We can plug any value into $g(x)$ ✓

We need $g(x) \geq 0$ (to take the square root of it)

This gives $x - 2 \geq 0$

$x \geq 2$

$\boxed{[2, \infty)}$

Example. Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 4$. Find the domain of $\left(\frac{f}{g}\right)(x)$. $= \dfrac{f(x)}{g(x)}$

For $f(x)$, domain is $x \neq 0$

For $g(x)$, domain is $(-\infty, \infty)$

We also need $g(x) \neq 0 \Rightarrow x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$

$\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)}$

Application: Calories Burned as a Function of Time

Fitness tracking often involves calculating the number of calories burned based on the time spent exercising. Suppose you know two things:

1. **Distance as a Function of Time:**
$$d(t) = 6t$$
   - Where t is the time in hours, and $d(t)$ is the distance (in miles) walked in that time.
2. **Calories Burned as a Function of Distance:**
$$C(d) = 100d$$
   - Where d is the distance (in miles), and $C(d)$ is the total calories burned.

We can use function **composition** to calculate the total calories burned as a function of time:

$$C(d(t)) = C(6t) = 100(6t) = 600t.$$

This composition directly links walking time to calories burned. For every hour of walking, the person burns 600 calories.