Limits

Definition. A **limit** describes the value a function approaches as the input approaches a specific value. In notation, we write

$$\lim_{x \to a} f(x) = L$$

to mean that as x gets closer and closer to a, the values of $f(x)$ get closer and closer to L.

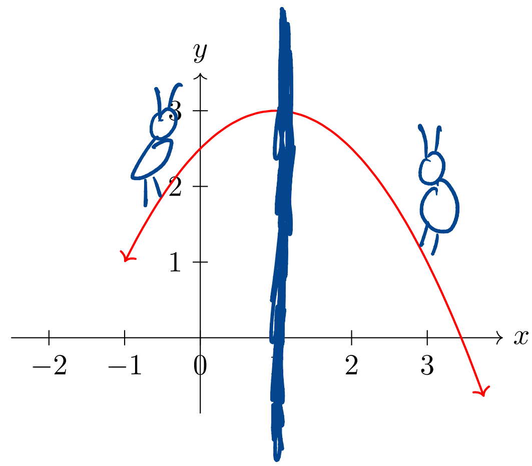
Example. Compute $\lim_{x \to 1} -\frac{1}{2}(x-1)^2 + 3$

Draw a line at $x=1$

Where would a bug hit the wall?

- From the left : 3
- From the right : 3

Hence $\lim_{x \to 1} f(x) = 3$



Note : The function is $\boxed{\text{continuous}}$ at $x=1$, so we also could have plugged in $x=1$ ... $f(1) = -\frac{1}{2}(x-1)^2 + 3 = 3$

Can't always do this

Definition. One-Sided Limits:

- The **left-hand limit** $\lim_{x \to a^-} f(x)$ is the value the function approaches as x approaches a from the left (that is, from values smaller than a).
- The **right-hand limit** $\lim_{x \to a^+} f(x)$ is the value the function approaches as x approaches a from the right (that is, from values larger than a).

Two-Sided Limits:

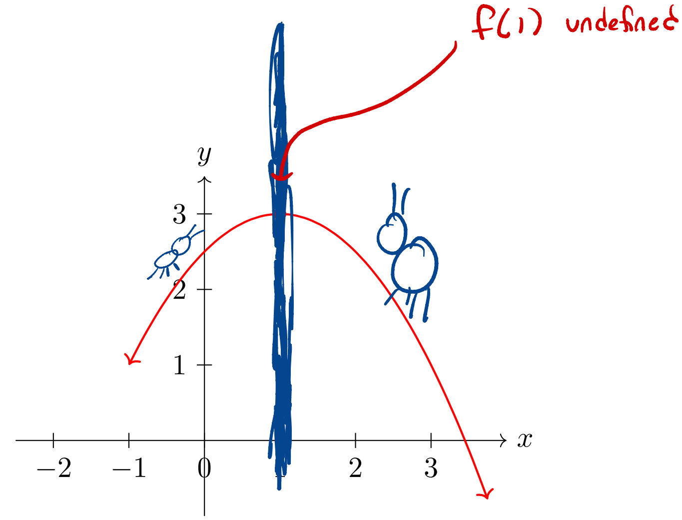
- The **two-sided limit** $\lim_{x \to a} f(x)$ exists if and only if both the left-hand and right-hand limits exist and are equal.

Example. Compute $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1^-} f(x) = 3$$

$$\lim_{x\to 1^+} f(x) = 3$$

$$\lim_{x\to 1} f(x) = 3$$



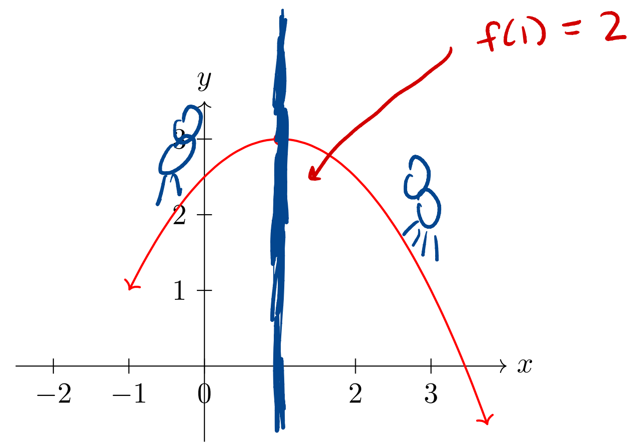
★ A limit can exist even if a function is not defined at that point

Example. Compute $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1^-} f(x) = 3$$

$$\lim_{x\to 1^+} f(x) = 3$$

$$\lim_{x\to 1} f(x) = 3$$



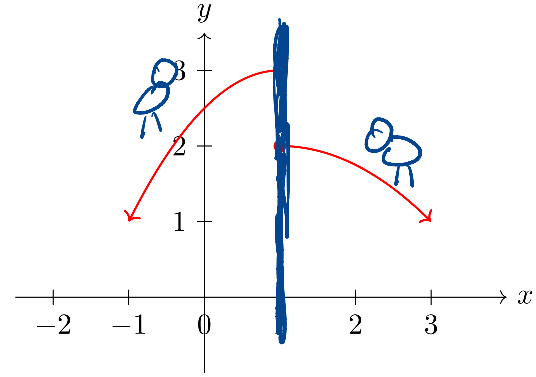
★ The value of the function at a point can be different than the limit

Example. Compute $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1^-} f(x) = 3$$

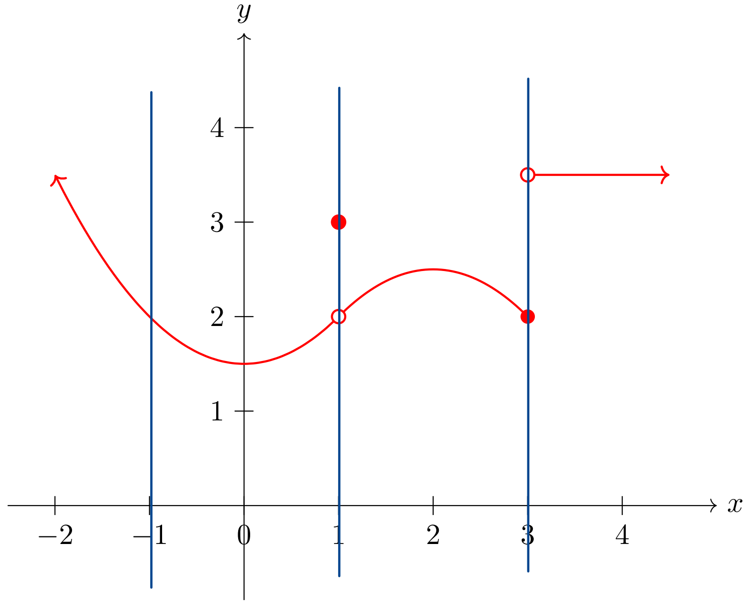
$$\lim_{x\to 1^+} f(x) = 2$$

$$\lim_{x\to 1} f(x) \text{ DNE}$$



★ A $\boxed{\text{jump discontinuity}}$ occurs when the left-hand and right-hand limits exist but are not equal. At a jump discontinuity, the two-sided limit does not exist

Example. The graph of a function $f(x)$ is shown below.



Answer the following:

$\boxed{x=-1}$ $\boxed{x=1}$ $\boxed{x=3}$

(a) $\lim_{x\to -1^-} f(x) =$ 2

(b) $\lim_{x\to -1^+} f(x) =$ 2

(c) $\lim_{x\to -1} f(x) =$ 2

(d) $f(-1) =$ 2

Continuous!

(e) $\lim_{x\to 1^-} f(x) =$ 2

(f) $\lim_{x\to 1^+} f(x) =$ 2

(g) $\lim_{x\to 1} f(x) =$ 2

(h) $f(1) =$ 3

limit is different than function value!

(i) $\lim_{x\to 3^-} f(x) =$ 2

(j) $\lim_{x\to 3^+} f(x) =$ 3.5

(k) $\lim_{x\to 3} f(x) =$ DNE

(l) $f(3) =$ 2

jump discontinuity!

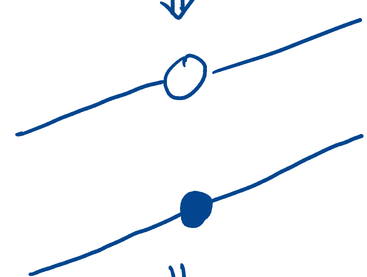
Example. Evaluate the limit by factoring:

$$\lim_{x\to 3} \frac{x^2-9}{x-3}.$$

The graph has a hole at 3. We factored and canceled the $(x-3)$ term and THEN computed the limit

$\Downarrow$

$\frac{x^2-9}{x-3}$



$x+3$

$\Downarrow$

Different functions at $x=3$, but same limit.

Factor the numerator:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)}$$

$$= \lim_{x\to 3} x+3$$

$$= 6$$

We can compute this by plugging in $x=3$, since $x+3$ is continuous

Example. Evaluate the limit by multiplying by the conjugate:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}.$$

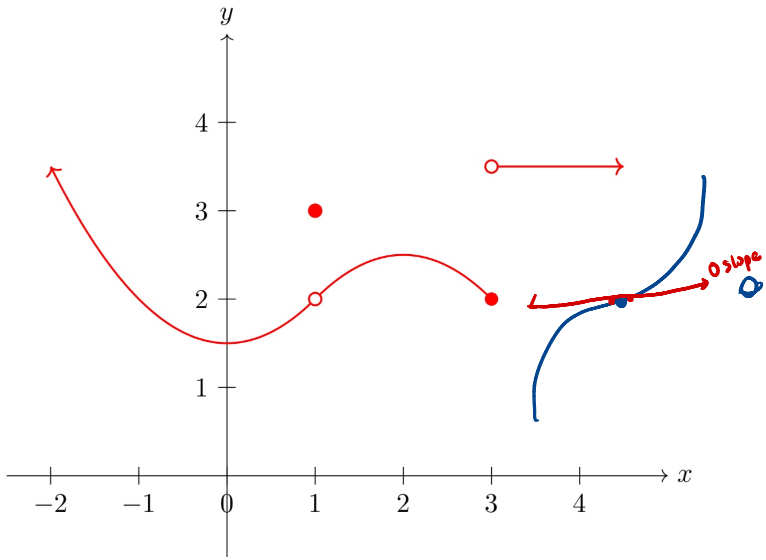
$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$

Now, plugging in 0 makes sense.

$$= \frac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$$



Q: what is the slope at x=2?