Half Angle Identities

Definition (Half Angle Identities).

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Remark. The sign of the square root depends on the quadrant in which $\frac{\theta}{2}$ lies.

Remark. You do not need to memorize the half angle identities. These identities will be listed on a provided formula sheet for the exam. You are responsible for memorizing the reciprocal, quotient, and Pythagorean identities.

Example. Given a right triangle where $\cos\theta = \frac{3}{5}$ and θ is in Quadrant I, evaluate $\sin\left(\frac{\theta}{2}\right)$.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

so $\sin\left(\frac{\theta}{2}\right) > 0$

Use the fact that $\frac{\theta}{2}$ is in quadrant I and also $\cos(\theta) = \frac{3}{5}$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-3/5}{2}} = \sqrt{\frac{2/5}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

Example. Evaluate $\cos(15°)$ using a half angle identity.

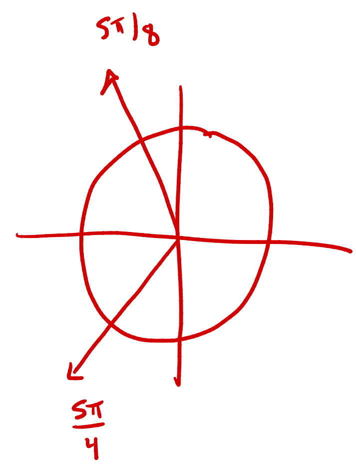
Note: $15° = \frac{30°}{2}$, so:

$$\cos(15°) = \cos\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{1+\cos(30°)}{2}}$$

Since $\cos(15°) > 0$ (it's in quadrant 1) and $\cos(30°) = \frac{\sqrt{3}}{2}$:

$$\cos(15°) = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example. Evaluate $\sin\left(\frac{5\pi}{8}\right)$ using a half angle identity.



Note: $\frac{5\pi}{8} = \frac{5\pi/4}{2}$

$$\sin\left(\frac{5\pi}{8}\right) = \sin\left(\frac{5\pi/4}{2}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{5\pi}{4}\right)}{2}}$$

Since $\sin\left(\frac{5\pi}{8}\right) > 0$ ($5\pi/8$ is in quadrant 2) and $\cos(5\pi/4) = -\frac{\sqrt{2}}{2}$:

$$\sin(5\pi/8) = \sqrt{\frac{1-(-\sqrt{2}/2)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example. Use a half-angle identity to solve the equation

$$\sin^2\left(\frac{x}{2}\right) = \frac{3}{4}, \qquad 0 \le x < 2\pi.$$

$$\sin^2\left(\frac{x}{2}\right) = \left(\sin\left(\frac{x}{2}\right)\right)^2 = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 = \frac{1-\cos x}{2}$$

Now solve:

$$\frac{1-\cos x}{2} = \frac{3}{4} \quad \Rightarrow \quad 1-\cos x = \frac{3}{2}$$

$$\Rightarrow \quad \cos x = -\frac{1}{2}$$

$$\Rightarrow \quad x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$