Double Angle Identities

Double angle identities allow us to express trigonometric functions of $2x$ in terms of functions of x. These identities are useful in simplifying expressions, solving equations, and evaluating trigonometric functions without a calculator.

Definition. The double angle identities for sine, cosine, and tangent are:

$$\sin(2x) = 2\sin(x)\cos(x)$$

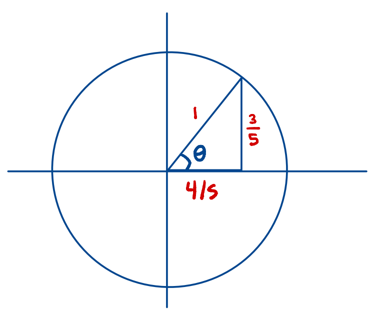
$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \quad \text{provided } \tan(x) \neq \pm 1$$

Remark. You do not need to memorize the double angle identities. These identities will be listed on a provided formula sheet for the exam. You are responsible for memorizing the reciprocal, quotient, and Pythagorean identities.

Example. Suppose θ is an angle in a right triangle such that $\sin(\theta) = \dfrac{3}{5}$ and θ is in Quadrant I. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

Since $\sin(\theta) = \frac{3}{5}$ and θ is in quadrant I, we have



$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3/5}{4/5} = \frac{3}{5}\cdot\frac{5}{4} = \frac{3}{4}$

So $\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{16/25} = \frac{4}{5}$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{2\cdot 3/4}{1-(3/4)^2} = \frac{24}{7}$$

Example. Given that $\cos(x) = -\frac{5}{13}$ and x is in Quadrant II, find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.

① Determine $\sin(x)$. Since x is in Quadrant II, $\sin(x) > 0$.

$$\sin^2(x) = 1 - \cos^2(x) = 1 - \left(-\frac{5}{13}\right)^2 = \frac{144}{169} \Rightarrow \sin(x) = \frac{12}{13}$$

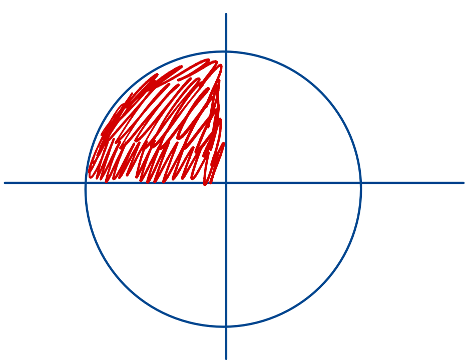
②

$$\sin(2x) = 2\sin(x)\cos(x) = 2 \cdot \frac{12}{13} \cdot \left(-\frac{5}{13}\right) = \frac{-120}{169}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \left(-\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = \frac{-119}{169}$$

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{-120/169}{-119/169} = \frac{120}{119}$$

Example. Suppose $\sin(2x) > 0$ and $\cos(2x) < 0$. In which quadrant does the angle $2x$ lie?



If $\sin(2x) > 0 \Rightarrow$ Quadrant I or II

If $\cos(2x) < 0 \Rightarrow$ Quadrant II or III

$2x$ is in quadrant II

Example. Solve the equation $\sin(x)\cos(x) = \frac{1}{4}$ for all solutions in the interval $[0, 2\pi)$.

Multiply both sides by 2

$$2\sin(x)\cos(x) = \frac{1}{2}$$

Double-angle identity:

$$\sin(2x) = \frac{1}{2}$$

Now solve

$$\sin(u) = \frac{1}{2}$$

$$\Rightarrow \quad u = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad u = \frac{5\pi}{6} + 2\pi n$$

$$\Rightarrow \quad 2x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2\pi n$$

$$\Rightarrow \quad x = \frac{\pi}{12} + \pi n \quad \text{or} \quad x = \frac{5\pi}{12} + \pi n$$

The solutions in $[0, 2\pi)$ are $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$