Functions

Introduction

Functions are one of the most powerful tools in mathematics, providing a way to model relationships and behaviors in the real world. This lecture will deepen your understanding of functions by exploring their definitions, domains, and ranges.

Definition of a Function

Definition. What is a function? A function is a relationship where each input corresponds to exactly one output

$$\text{input } x \longrightarrow \boxed{\begin{matrix}\text{function}\\ f\end{matrix}} \longrightarrow \text{output } f(x)$$

<u>Notation:</u>

x : input (independent variable)

f : function f takes input f

$f(x)$: output

$f(\text{input}) = 2 \cdot \text{input} + 3$

Example. Let $f(x) = 2x + 3$. Find $f(4)$.

If $x=4$, then $f(4) = 2 \cdot 4 + 3 = 11$

Example. Evaluate $f(x) = x^2 - 5x + 6$ for $x = 3$.

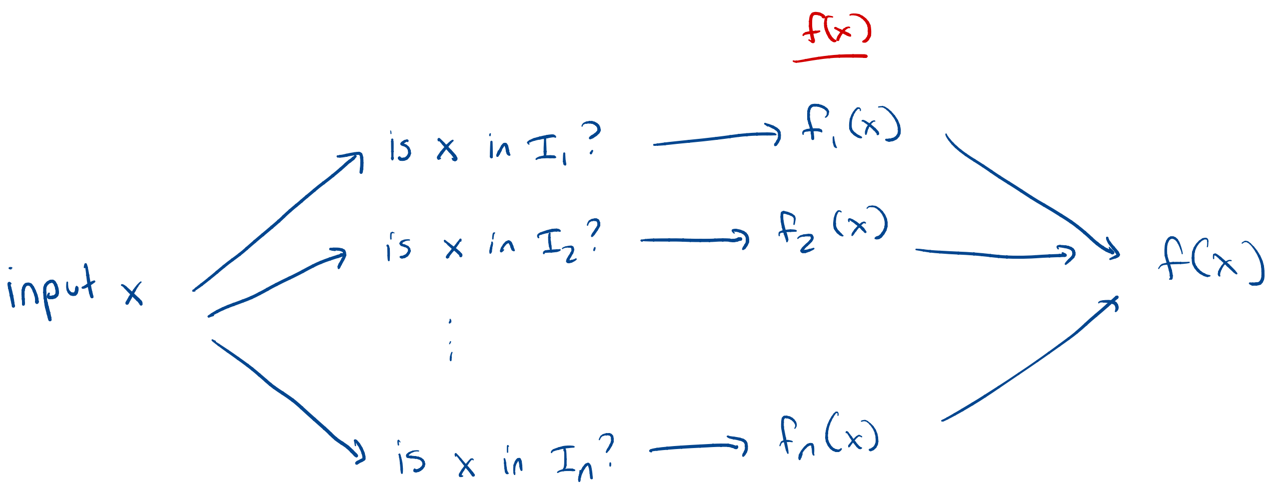
If $x=3$, then
$$\begin{aligned} f(3) &= 3^2 - 5 \cdot 3 + 6 \\ &= 9 - 15 + 6 \\ &= 0 \end{aligned}$$

Definition. A **piecewise function** is a function described by multiple subfunctions,

(input values)
each applying to a specific interval of the domain:

$$f(x) = \begin{cases} f_1(x), & \text{if } x \text{ is in } I_1, \\ f_2(x), & \text{if } x \text{ is in } I_2, \\ \vdots & \\ f_n(x), & \text{if } x \text{ is in } I_n. \end{cases}$$

Here, $f_1(x), f_2(x), \dots, f_n(x)$ are the subfunctions and $I_1, \dots, I_n$ are the corresponding intervals for the inputs.



Example. Evaluate the piecewise function:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -2x & \text{if } x < 0. \end{cases}$$

If $x = 4$, then $f(4) = 4^2 = 16$

If $x = -3$, then $f(-3) = -2 \cdot (-3) = 6$

Domain and Range

Definition. The domain of a function $f(x)$ is the set of all inputs for which the function is defined.

Definition. The range of a function $f(x)$ is the set of all outputs the function can produce.

Example.

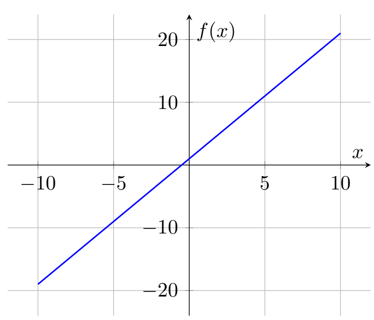
Function Type	Domain	Range
Linear Function: $f(x) = mx + b$	$(-\infty, \infty)$	$(-\infty, \infty)$
Quadratic Function: $f(x) = ax^2 + bx + c$	$(-\infty, \infty)$	Determined by the vertex: $a>0$: $[k, \infty)$ $a<0$: $(-\infty, k]$
Square Root Function: $f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$

the parabola points up

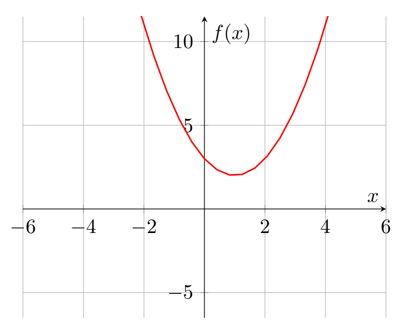
the parabola points down

We cannot take the square root of a negative number

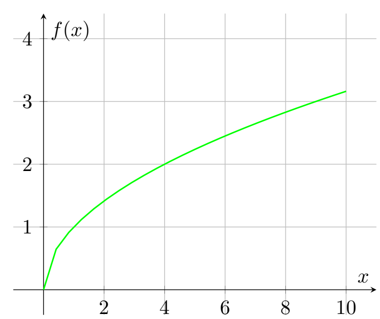
Linear Function



Quadratic Function



Square Root Function



Example. Find the domain and range of $f(x) = \sqrt{x+2}$

Typically you can set up an inequality to find your domain/range

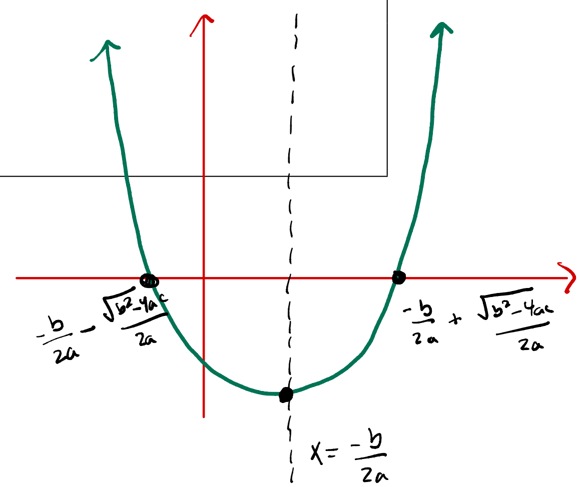
Domain: We need $x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow \boxed{[-2, \infty)}$

Range: Taking the square root of something can only produce values 0 and above. $\boxed{[0, \infty)}$

Quadratic Formula: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Quadratic Functions and the Vertex Formula

Vertex Formula for Quadratic Functions
For $f(x) = ax^2 + bx + c$, the vertex is given by: $$x = -\frac{b}{2a}, \quad f(x) = f\left(-\frac{b}{2a}\right).$$



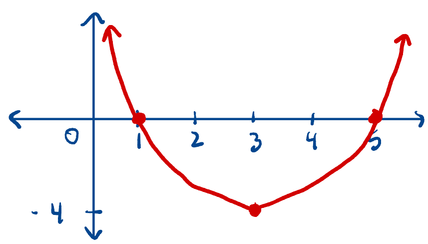
Example: Find the vertex of $f(x) = x^2 - 6x + 5$.

① The vertex is at $x = \frac{-b}{2a} = \frac{-(-6)}{2 \cdot 1} = 3$

The y-coord. is $f(3) = 3^2 - 6(3) + 5 = -4$

$\Rightarrow$ The vertex is $(3, -4)$

② Suppose we wanted to graph $f(x)$.



Where is $f(x) = 0$?

We need to solve $x^2 - 6x + 5 = 0$

$\Rightarrow (x-5)(x-1) = 0$

$\Rightarrow x = 5$ or $x = 1$

Example. Find the range of $f(x) = x^2 - 4$.

Vertex: The x-coord is $x = \frac{-b}{2a} = \frac{0}{2 \cdot 1} = 0$

The y-coord is $f(0) = 0^2 - 4 = -4$

The range is $[-4, \infty)$ because the parabola points up

because $a > 0$

The Difference Quotient

Definition. The difference quotient is a measure of the average rate of change of a function over an interval. It represents the slope of the secant line between two points on the graph of a function $f(x)$ separated by a horizontal distance h:

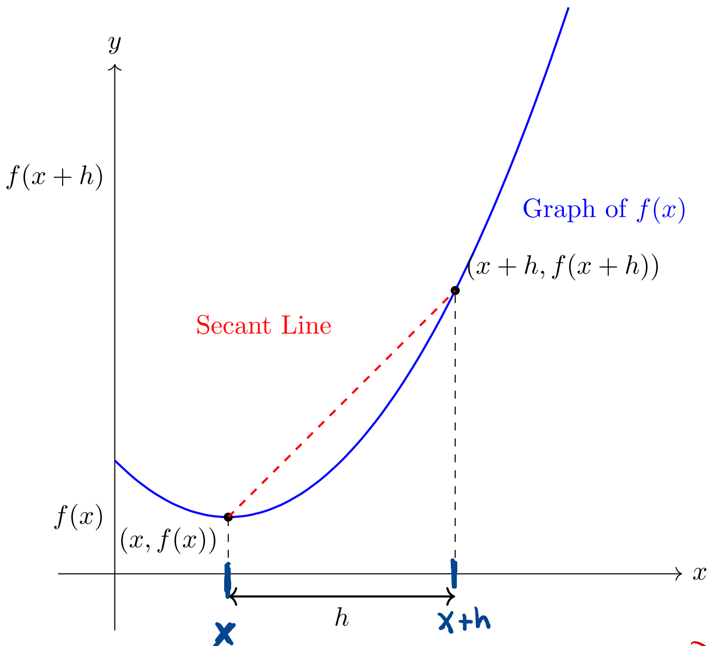
$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0.$$

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$

rise: $f(x+h) - f(x)$

run: h

$$\text{Slope} = \frac{f(x+h)-f(x)}{h}$$



The difference quotient is studied extensively in calculus.

Example. Compute the difference quotient for $f(x) = x^2$

$f(\text{input}) = \text{input}^2$

Common exam question is to compute the difference quotient of a function

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x+h$$

Example. Compute the difference quotient for $f(x) = 3x - 4$

$$\frac{f(x+h)-f(x)}{h} = \frac{\left[3(x+h)-4\right]-\left[3x-4\right]}{h}$$

$$= \frac{\cancel{3x}+3h\cancel{-4}-\cancel{3x}\cancel{+4}}{h} = \frac{3h}{h} = 3$$

Applications

Example. The drug dosage $D(w)$ that a doctor prescribes is a function of the patient's weight w.

Example. The amount of money earned $E(h)$ is a function of the number of hours h worked.