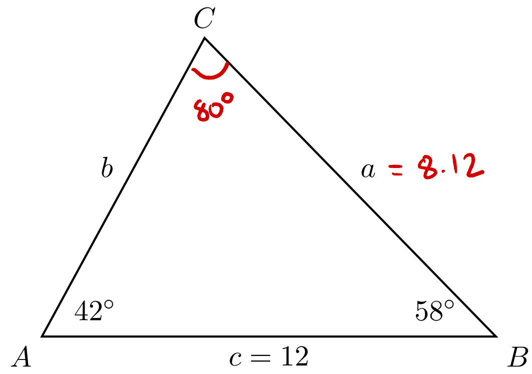
Laws of Sines and Cosines

Theorem (Law of Sines)**.** In any triangle with angles A, B, and C, and opposite sides a, b, and c, respectively, the Law of Sines states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example. Use the Law of Sines to find all sides and angles of the given triangle.



① Find C: $C + 42° + 58° = 180°$

$C = 80°$

② $\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin(42°)}{a} = \frac{\sin(80°)}{12}$

$a = 12 \cdot \frac{\sin(42°)}{\sin(80°)} \approx 8.12$

③ $\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin(58°)}{b} = \frac{\sin(80°)}{12}$

$b = 12 \cdot \frac{\sin(58°)}{\sin(80°)} \approx 10.20$

Theorem (Law of Cosines)**.** In any triangle with sides a, b, and c, and corresponding opposite angles A, B, and C, the Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

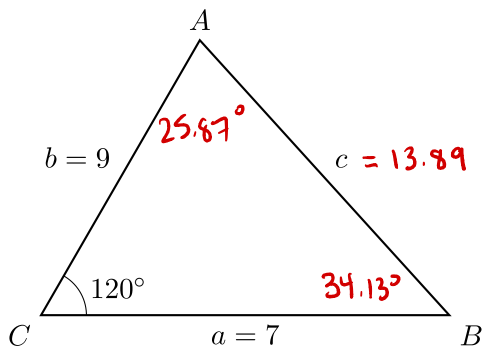
This is a generalization of the pythagorean thm.

If $C = 90° \Rightarrow c^2 = a^2 + b^2$

This can be rewritten for other sides:

$$\boxed{a^2} = b^2 + c^2 - 2bc\cos \boxed{A} \quad \text{and} \quad \boxed{b^2} = a^2 + c^2 - 2ac\cos \boxed{B}$$

Example. Use the Law of Cosines to find all sides and angles of the given triangle.



① $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 7^2 + 9^2 - 2\cdot 7\cdot 9\cos 120°$

$c^2 = 49 + 81 - 126\left(-\frac{1}{2}\right)$

$c^2 = 193$

$c = \sqrt{193} \approx 13.89$

② $b^2 = a^2 + c^2 - 2ac\cos B$

$9^2 = 7^2 + (\sqrt{193})^2 - 2\cdot 7\cdot\sqrt{193}\cos B$

$81 = 49 + 193 - 14\sqrt{193}\cos B$

$-161 = -14\sqrt{193}\cos B$

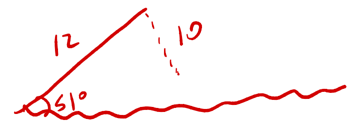
$\cos B = \dfrac{-161}{-14\sqrt{193}} = 0.828$

$B = \cos^{-1}(0.828) = 34.13°$

③ $A + 34.13° + 120° = 180°$

$A = 25.87°$

Fishing Pole

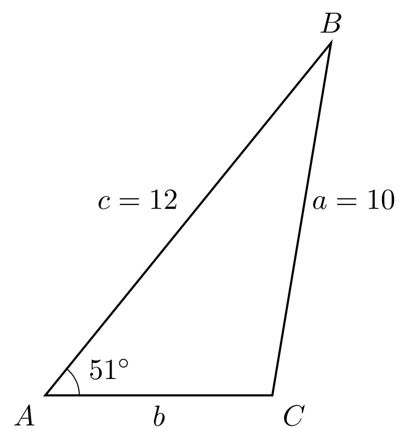


Theorem (Ambiguous Case for the Law of Sines)**.** When using the Law of Sines to solve for an angle, applying the inverse sine function may lead to *two possible angle values*, resulting in what is known as the **ambiguous case**.

Possible outcomes:

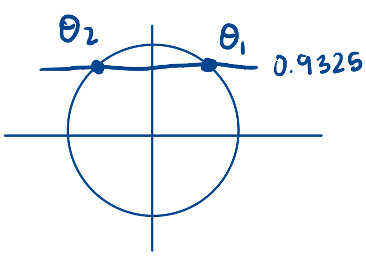
- **Two triangles:** If both θ and $180° - \theta$ lead to valid triangle angle measures, then two distinct triangles are possible.
- **One triangle:** If only one of the two angle values leads to a valid triangle, then exactly one triangle is possible.
- **No triangle:** If $\sin\theta > 1$ or neither angle produces valid measures, then no triangle can be formed.

Example. Find all sides and angles of the given triangle.



$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin(51°)}{10} = \frac{\sin C}{12}$$

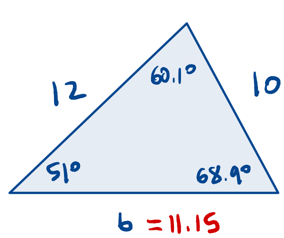
$$\Rightarrow \sin C = \frac{12}{10} \cdot \sin(51°) \approx 0.9325$$



Both θ_1 and θ_2 are possible angles in a triangle

① $C = \sin^{-1}(0.9325) = 68.9°$

$B = 180° - 51° - 68.9° = 60.1°$

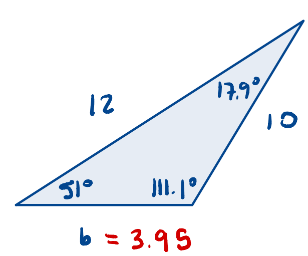


$b^2 = 10^2 + 12^2 - 2 \cdot 10 \cdot 12 \cos(60.1°)$

$b = 11.15$

② $C = 180° - \sin^{-1}(0.9325) = 111.1°$

$B = 180° - 51° - 111.1° = 17.9°$



$b^2 = 10^2 + 12^2 - 2 \cdot 10 \cdot 12 \cos(17.9°)$

$b = 3.95$