Inverse Trigonometric Identities

Definition. The inverse sine function $\sin^{-1}(x)$, also written as $\arcsin(x)$, is the angle θ such that $\sin(\theta) = x$, where θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$$\text{Domain: } -1 \leq x \leq 1 \qquad \text{Range: } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Definition. The inverse cosine function $\cos^{-1}(x)$, also written as $\arccos(x)$, is the angle θ such that $\cos(\theta) = x$, where θ is between 0 and π.

$$\text{Domain: } -1 \leq x \leq 1 \qquad \text{Range: } 0 \leq \theta \leq \pi$$

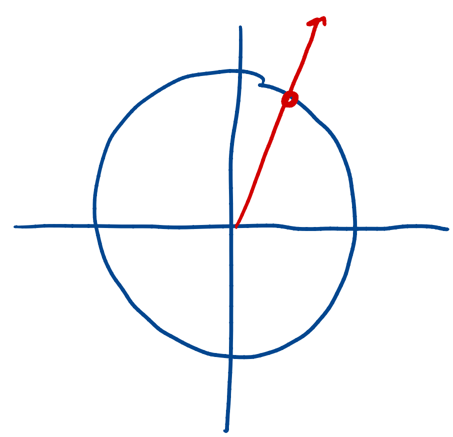
Definition. The inverse tangent function $\tan^{-1}(x)$, also written as $\arctan(x)$, is the angle θ such that $\tan(\theta) = x$, where θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (but not equal to $\pm\frac{\pi}{2}$).

$$\text{Domain: all real numbers } x \in \mathbb{R} \qquad \text{Range:} -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Example. Evaluate $\cos^{-1}(\cos(\frac{\pi}{3}))$

$$\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

"The angle in $[0, \pi]$ whose cosine is $\frac{1}{2}$"

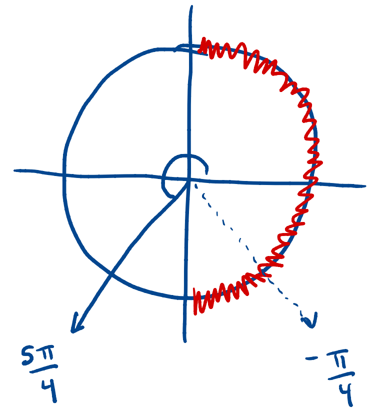


We got back what we started with, since $\frac{\pi}{3}$ is in $[0, \pi]$

Example. Evaluate $\sin^{-1}(\sin(\frac{5\pi}{4}))$

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-\frac{\pi}{4}}$$

We did not get back what we started with, since $\frac{5\pi}{4}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



True/False: For any θ in $(-\infty, \infty)$, the following identity holds: $\sin^{-1}(\sin(\theta)) = \theta$.

arcsin outputs an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So if θ is outside of this interval, we WON'T get back what we started with.

True/False: For any x in $[-1, 1]$, the following identity holds: $\sin(\sin^{-1}(x)) = x$.

$\sin^{-1}(x)$ gives you an angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

So... if you take the sine of this θ, you get x.

Analogy:

Let $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$

$f(f^{-1}(x)) = (\sqrt{x})^2$ is always x, but

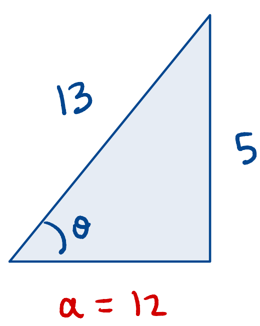
$f^{-1}(f(x)) = \sqrt{x^2}$ is not always x

$$2 \xrightarrow{f} 4 \xrightarrow{f^{-1}} 2 \quad \checkmark$$

$$-2 \xrightarrow{f} 4 \xrightarrow{f^{-1}} 2 \quad \times$$

Example. Evaluate $\cos\left(\sin^{-1}\left(\frac{5}{13}\right)\right)$

If x is not a standard value on the unit circle, we can use triangles



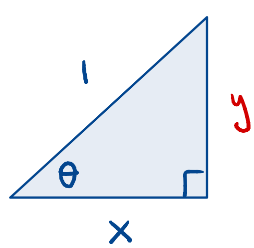
- Draw the triangle so that $\theta = \sin^{-1}\left(\frac{5}{13}\right)$
- Solve the triangle: $a = \sqrt{13^2 - 5^2} = 12$
- $\cos\theta = \dfrac{a}{h} = \dfrac{12}{13}$

Example. Rewrite the expression as an algebraic expression in terms of x:

$$\tan\left(\cos^{-1}(x)\right)$$

Can do this with reference triangles

⚠ As long as you know this only illustrates the case θ in $[0, \frac{\pi}{2}]$ even though θ can be any angle in $[0, \pi]$

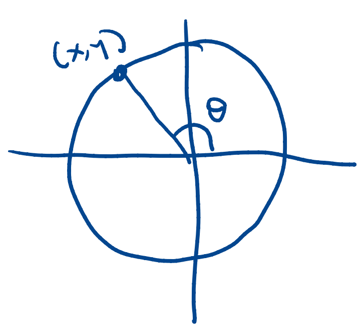


Draw θ so that $\theta = \cos^{-1}(x)$

Solve for y: $y = \sqrt{1 - x^2}$

$\tan(\theta) = \dfrac{o}{a} = \dfrac{\sqrt{1-x^2}}{x}$

How I would do this:



$\cos^{-1}(x)$ is the angle θ in $[0, \pi]$ with x-coord x. Then the y-coord is $y = \sqrt{1-x^2}$, since we are on the unit circle. Hence $\tan(\theta) = \dfrac{y}{x} = \dfrac{\sqrt{1-x^2}}{x}$

$x^2 + y^2 = 1$