Inverse Trigonometric Functions

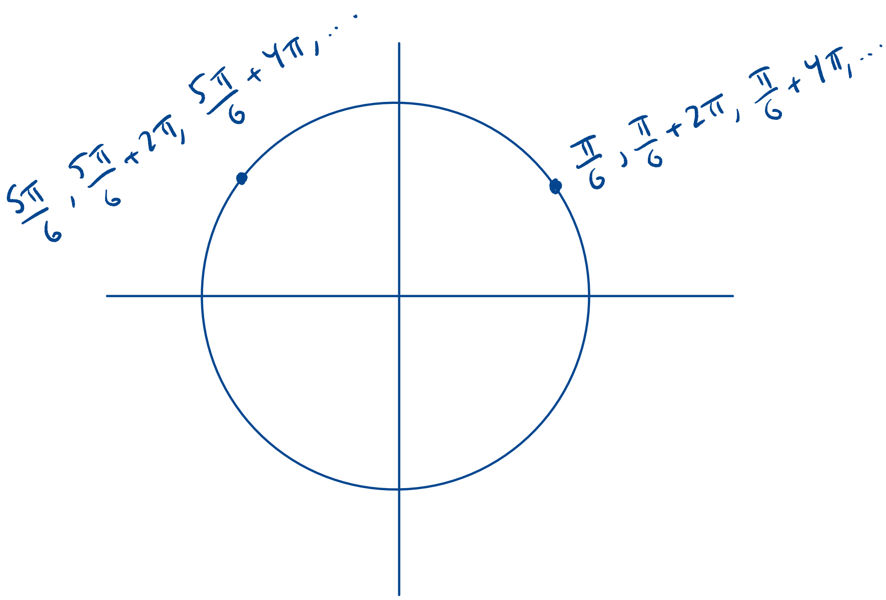
The trig functions $\sin(\theta)$ and $\cos(\theta)$ take an angle and tell you a point on the unit circle:

$$\sin(\theta) \text{ is the } y\text{-coordinate of the point at angle } \theta$$

$$\cos(\theta) \text{ is the } x\text{-coordinate of the point at angle } \theta$$

Question: Can we define functions to go the other way? Given an x- or y-coordinate on the unit circle, can we output the angle θ on the unit circle with that particular x- or y-coordinate?

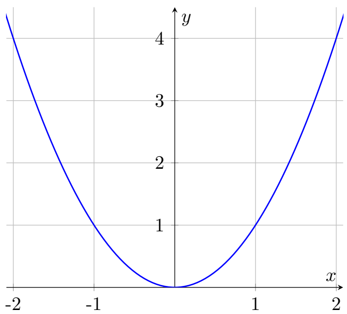
Issue: There are infinitely many angles with a particular x- or y-coordinate. To define a function, we can only have one output.



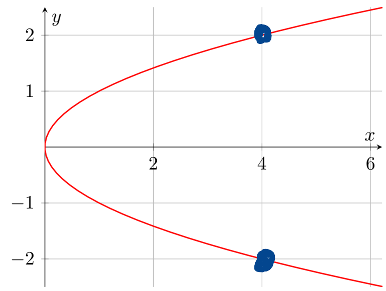
If we want θ so that $\sin\theta = \frac{1}{2}$, how do we decide which θ to choose?

There are infinitely many θ with $\sin\theta = \frac{1}{2}$, but to define a function, we must choose <u>one</u>.

How we answered this question with $f(x) = x^2$



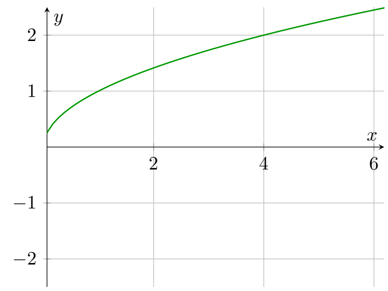
$f(x) = x^2$



This is what the "actual" inverse should look like

(right? 4 comes from both -2 and 2)

Issue: <u>NOT</u> a function!



<u>Fix</u>: by convention, we declare the square root FUNCTION to choose the <u>positive</u> square root.

<u>IS</u> a function!

⚠ Things get wacky.

$$2 \xrightarrow{f(x)=x^2} 4 \xrightarrow{f^{-1}(x)=\sqrt{x}} 2 \quad \checkmark$$

$$-2 \xrightarrow{f(x)=x^2} 4 \xrightarrow{f^{-1}(x)=\sqrt{x}} 2 \quad \times$$

$$\Downarrow$$

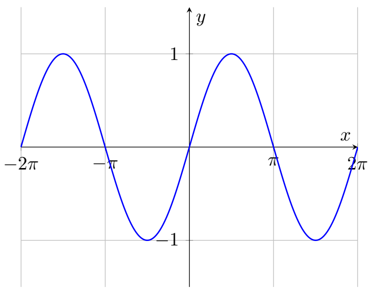
$f^{-1}(f(2)) = 2$ ... inverse! ✓

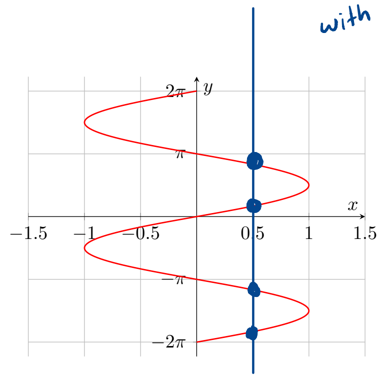
$f^{-1}(f(-2)) = 2$ ... weird!

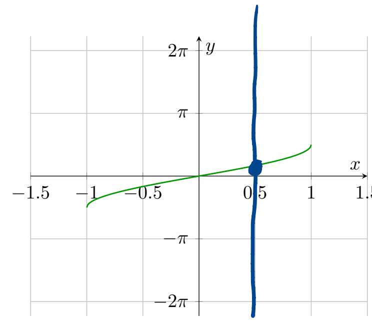
The inverse of $\sin(x)$

outputs the angles with $\sin\theta = \frac{1}{2}$

only 1 output!





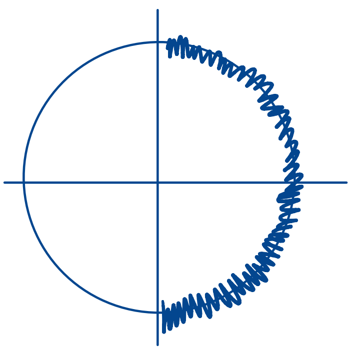


$f(x) = \sin x$

This is what the inverse "should" look like, except it is not a function

Fix: find an interval with no repeats...

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ works!



Big idea: every angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has a unique y-coordinate and we cover all possibilities!

(we could have chosen $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, etc...)

Definition. The **inverse sine function**, written as $\arcsin(x)$ or $\sin^{-1}(x)$, tells you the angle whose sine is x.

We only output angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because that is an interval where the sine graph doesn't repeat.

Question. What is the domain of $\arcsin(x)$?

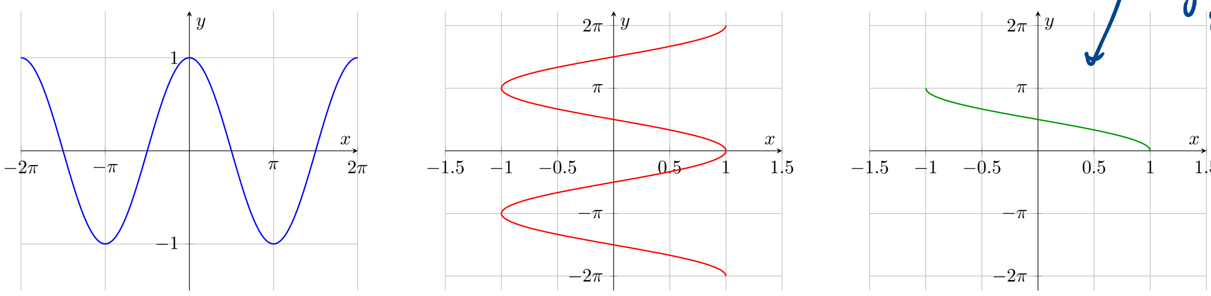
$[-1, 1]$ ← All y-coords are between −1 and 1

Question. What is the range of $\arcsin(x)$?

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ← We output the unique θ with the given y-coordinate

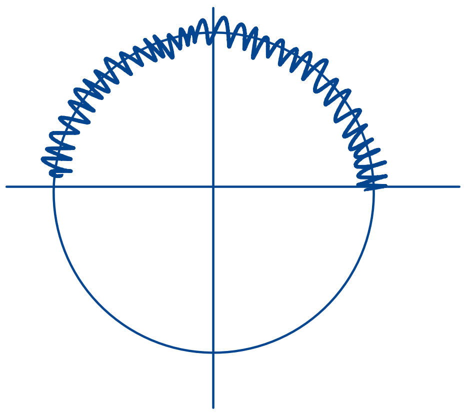
The inverse of $\cos(x)$

output an angle between 0 and π



cos(x)

arccos(x)



Similarly, angles in $[0, \pi]$ have unique x-coordinates and we cover all possibilities.

Definition. The **inverse cosine function**, written as $\arccos(x)$ or $\cos^{-1}(x)$, tells you the angle whose cosine is x.

We only output angles between 0 and π, because that is an interval where the cosine graph doesn't repeat.

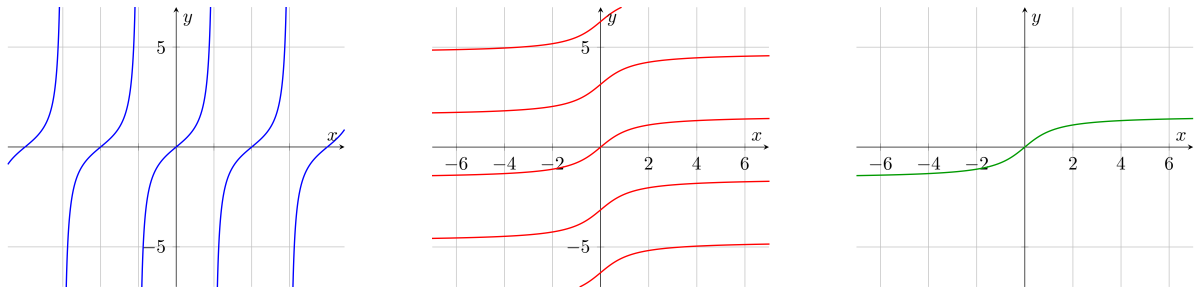
Question. What is the domain of $\arccos(x)$?

$[-1, 1]$ ← The possible values of x-coordinates

Question. What is the range of $\arccos(x)$?

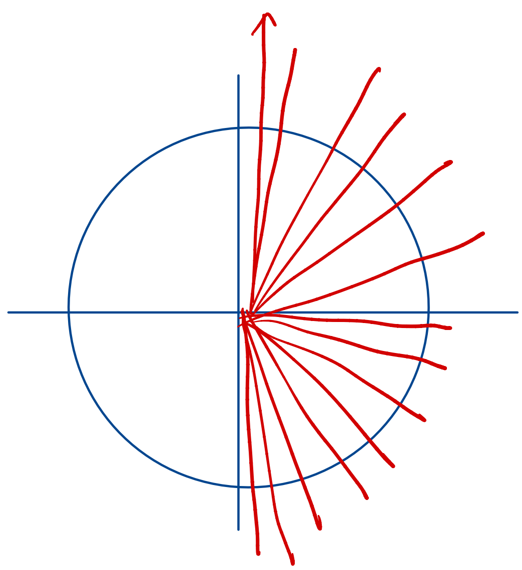
$[0, \pi]$ ← we output the <u>unique</u> angle with the given x-coordinate

The inverse of $\tan(x)$



tan(x)

arctan(x)



Angles in $(-\frac{\pi}{2}, \frac{\pi}{2})$ have unique slopes and cover all possibilities

(Note: $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ have undefined tangent, so this is an open interval)

Definition. The **inverse tangent function**, written as $\arctan(x)$ or $\tan^{-1}(x)$, tells you the angle whose ~~sine~~ tangent is x.

We only output angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because that is an interval where the tangent graph doesn't repeat.

Question. What is the domain of $\arctan(x)$?

$(-\infty, \infty)$ ← the slope can be anything!

Question. What is the range of $\arctan(x)$?

$(-\frac{\pi}{2}, \frac{\pi}{2})$ ← output the unique θ with the given slope.