Solving Trigonometric Equations

Methods for Solving Equations

Definition. Common methods for solving trigonometric equations include:

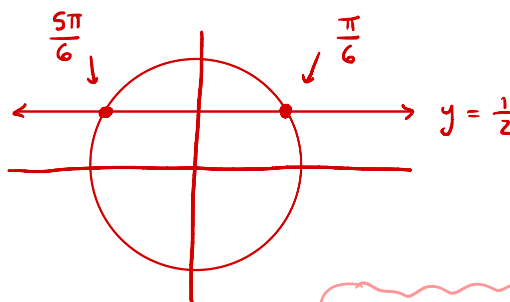
- Linear methods
- Quadratic methods
- Factoring
- u-substitution

Important: Do not divide both sides by a factor, as this could eliminate solutions.

Linear Methods

Example Without Domain Restrictions

Example. Solve $2\sin\theta - 1 = 0$.



$2\sin\theta - 1 = 0$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

The general solution is

$\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for all integers k.

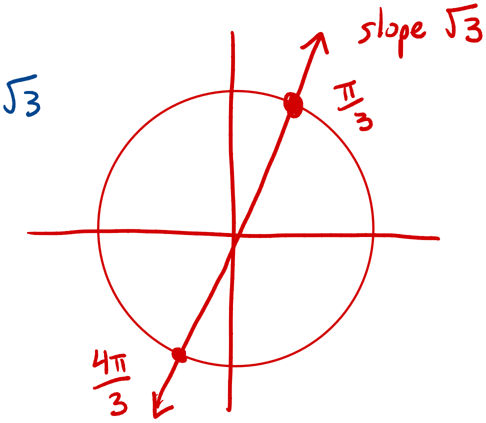
1. Find the solutions within $[0, 2\pi]$
2. Add $2\pi k$ to these to get all solutions

Example With Domain Restrictions

Example. Solve $\tan\theta = \sqrt{3}$ on $0 \le \theta < 2\pi$.

Check: $\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$

$\theta = \frac{\pi}{3}$ or $\theta = \frac{4\pi}{3}$



Example With an Extraneous Solution

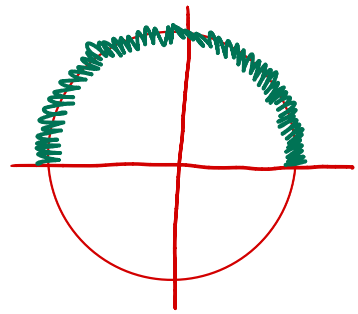
Example. Solve $\sin\theta = \sqrt{1-\cos^2\theta}$, checking for extraneous solutions.

Square both sides (carefully)

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

$\Rightarrow$ This is true for all angles θ



However, the original equation implies $\sin\theta > 0$

Thus θ has to be in quadrants I or II

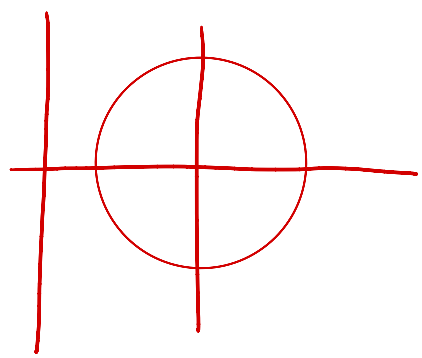
Example With No Solution

Example. Solve $2\cos\theta + 3 = 0$.

$$2\cos\theta + 3 = 0$$

$$2\cos\theta = -3$$

$$\cos\theta = -\frac{3}{2}$$



$x = -\frac{3}{2}$

No solution since $-1 \leq \cos\theta \leq 1$

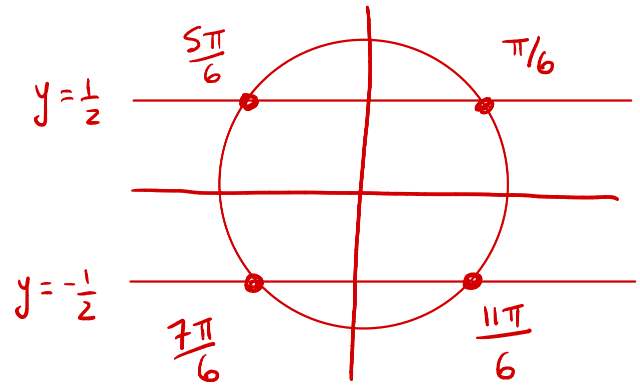
Quadratic Methods

Example. Solve $4\sin^2(\theta) - 1 = 0$.

$$4\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{4}$$

$$\sin\theta = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$



$$\boxed{\theta = \frac{\pi}{6} + 2\pi k, \quad \frac{5\pi}{6} + 2\pi k, \quad \frac{7\pi}{6} + 2\pi k, \quad \frac{11\pi}{6} + 2\pi k}$$

if you want, can simplify to $\theta = \frac{\pi}{6} + \pi k$ or $\frac{5\pi}{6} + \pi k$

Example. Solve $2\cos^2(\theta) - 3\cos(\theta) + 1 = 0$.

Let $y = \cos(\theta)$

$$2y^2 - 3y + 1 = 0$$

Need two numbers that multiply to $2 \cdot 1$ and add to -3

$$2y^2 - 2y - y + 1 = 0$$

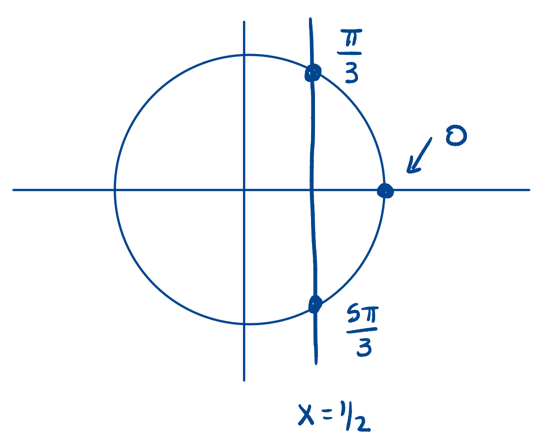
Factor by grouping

$$2y(y-1) - 1(y-1) = 0$$

$$(2y-1)(y-1) = 0$$

$$y = \frac{1}{2} \quad \text{or} \quad y = 1$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$



The general solution is $\boxed{\theta = \frac{\pi}{3} + 2\pi k, \quad \frac{5\pi}{3} + 2\pi k, \quad 0 + 2\pi k}$ for all integers k

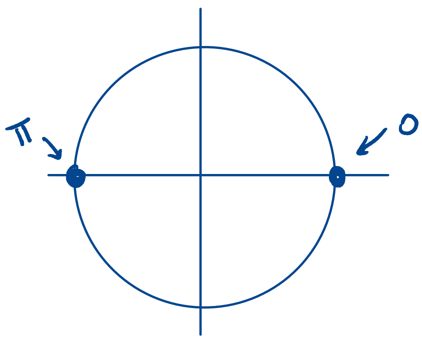
Factoring Methods

Example. Solve $\sin(\theta)\cos(\theta) - \sin(\theta) = 0$.

Factoring, we get: $\sin\theta\,(\cos\theta - 1) = 0$

Set each factor equal to 0: $\sin\theta = 0$ or $\cos\theta - 1 = 0$

$\cos\theta = 1$

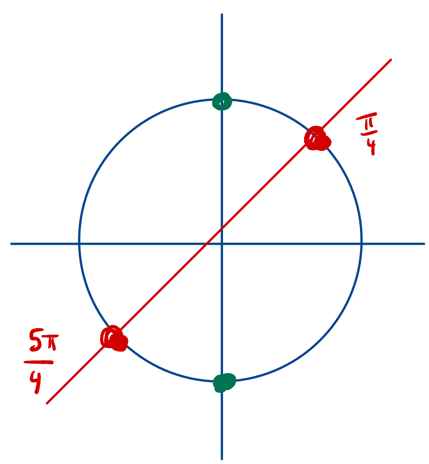


The general solution is $\theta = \boxed{0 + 2\pi k \quad \text{or} \quad \pi + 2\pi k}$

Alternatively, we can also write $\theta = \boxed{\pi k}$

Example. Solve $\cot^2(\theta) - \cot(\theta) = 0$.

$(\cot\theta)(\cot\theta - 1) = 0$



$\cot\theta = 0$ or $\cot\theta - 1 = 0$

$\cot\theta = 1$

$\Rightarrow \dfrac{\cos\theta}{\sin\theta} = 0$

$\Rightarrow \dfrac{\cos\theta}{\sin\theta} = 1$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \cos\theta = \sin\theta$

$\boxed{\theta = \frac{\pi}{2} + \pi k}$

$\boxed{\theta = \frac{\pi}{4} + \pi k}$

The general solution is $\boxed{\theta = \frac{\pi}{4} + \pi k \quad \text{or} \quad \frac{\pi}{2} + \pi k}$ for integers k

u-Substitution Methods

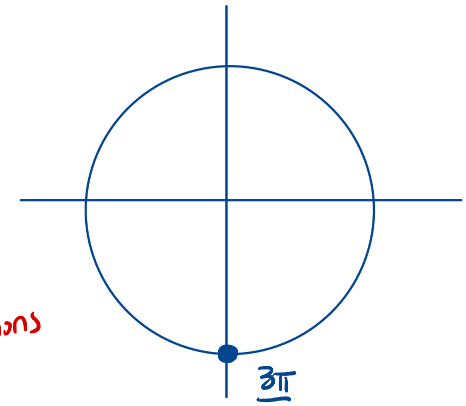
Example. Solve $\sin\left(\frac{\theta}{3}\right) = -1$.

Let $u = \frac{\theta}{3}$

Then we solve $\sin(u) = -1$

$$u = \frac{3\pi}{2} + 2\pi k$$

k is the # of rotations to add



$\frac{3\pi}{2} + 2\pi$

$\frac{3\pi}{2} + 4\pi$

$\frac{3\pi}{2} + 6\pi$

Substituting back,

$$\frac{\theta}{3} = \frac{3\pi}{2} + 2\pi k$$

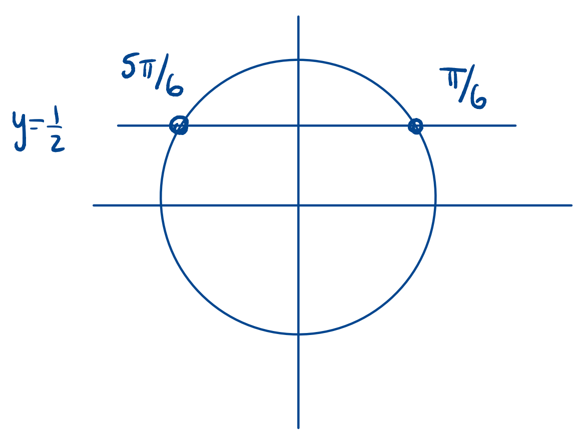
$$\theta = 3\left(\frac{3\pi}{2} + 2\pi k\right)$$

$$\boxed{\theta = \frac{9\pi}{2} + 6\pi k}$$

Example. Solve $\sin(2\theta) = \frac{1}{2}$ on $0 \le \theta < \pi$.

Let $u = 2\theta$

Then $\sin(u) = \frac{1}{2}$



$$u = \frac{\pi}{6} + 2\pi k \quad \text{or} \quad \frac{5\pi}{6} + 2\pi k$$

$$2\theta = \frac{\pi}{6} + 2\pi k \quad \text{or} \quad 2\theta = \frac{5\pi}{6} + 2\pi k$$

$$\theta = \frac{\pi}{12} + \pi k \quad \text{or} \quad \theta = \frac{5\pi}{12} + \pi k$$

Now we check k values to ensure $0 \le \theta < \pi$: $\boxed{\theta = \frac{\pi}{12}, \frac{5\pi}{12}}$