Trigonometric Identities

This section covers fundamental trigonometric identities: the Pythagorean, reciprocal, quotient, even/odd, and cofunction identities.

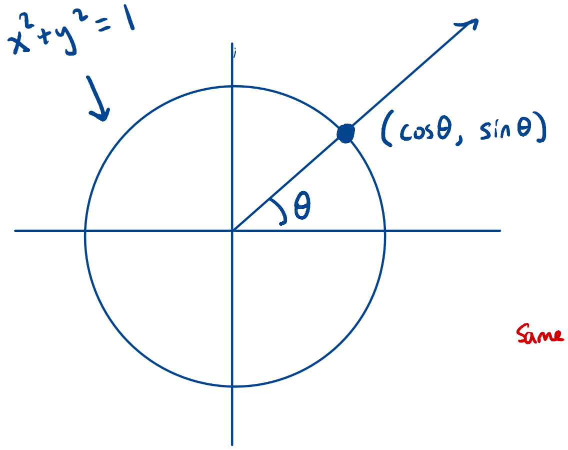
Pythagorean Identities

Definition. The fundamental identity derived from the unit circle $x^2 + y^2 = 1$ is:

$$\sin^2\theta + \cos^2\theta = 1.$$

From this, we obtain two additional identities:

$$1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta = \csc^2\theta.$$



For all θ, the point $(\cos\theta, \sin\theta)$ is on the unit circle. So

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Same thing

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Reciprocal and Quotient Identities

Definition (Reciprocal Identities).

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

This is only true when $\tan\theta$ is defined. $\tan\theta$ is undefined at $\frac{\pi}{2} + \pi k$

For this reason, we define $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

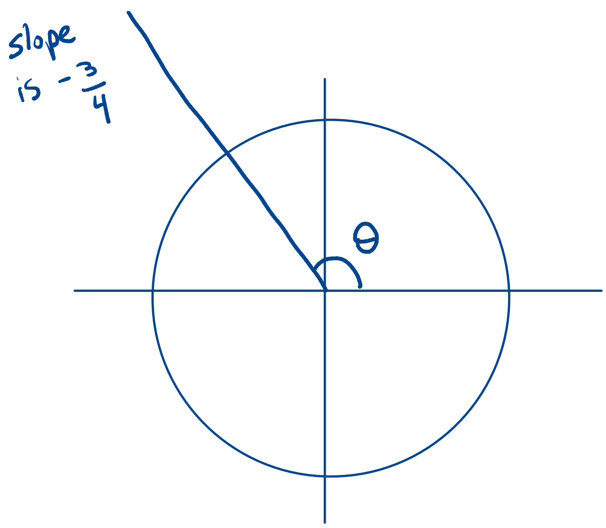
Definition (Quotient Identities).

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Note: $\cot(\frac{\pi}{2}) = 0$ but $\tan(\frac{\pi}{2})$ D.N.E.!

Using Identities in Different Quadrants

Example. Given $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find $\sin\theta$ and $\cos\theta$.



① $1 + \tan^2\theta = \sec^2\theta$

$1 + \left(-\frac{3}{4}\right)^2 = \sec^2\theta$

$1 + \frac{9}{16} = \frac{1}{\cos^2\theta}$

$\frac{25}{16} = \frac{1}{\cos^2\theta}$

$\cos^2\theta = \frac{16}{25} \Rightarrow \cos\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

Since $\cos\theta < 0$ (θ in Quadrant II), $\boxed{\cos\theta = -\frac{4}{5}}$

② $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{-4}{5}\right)^2 = 1$

$\sin^2\theta + \frac{16}{25} = 1 \Rightarrow \sin^2\theta = \frac{9}{25} \Rightarrow \sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$

$\sin\theta > 0 \Rightarrow \boxed{\sin\theta = \frac{3}{5}}$

Even/Odd and Cofunction Identities

odd: $f(-x) = -f(x)$

even: $f(-x) = f(x)$

Definition (Even/Odd Identities).

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \sec(-\theta) = \sec\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \cot(-\theta) = -\cot\theta$$

Definition (Cofunction Identities).

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$
$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

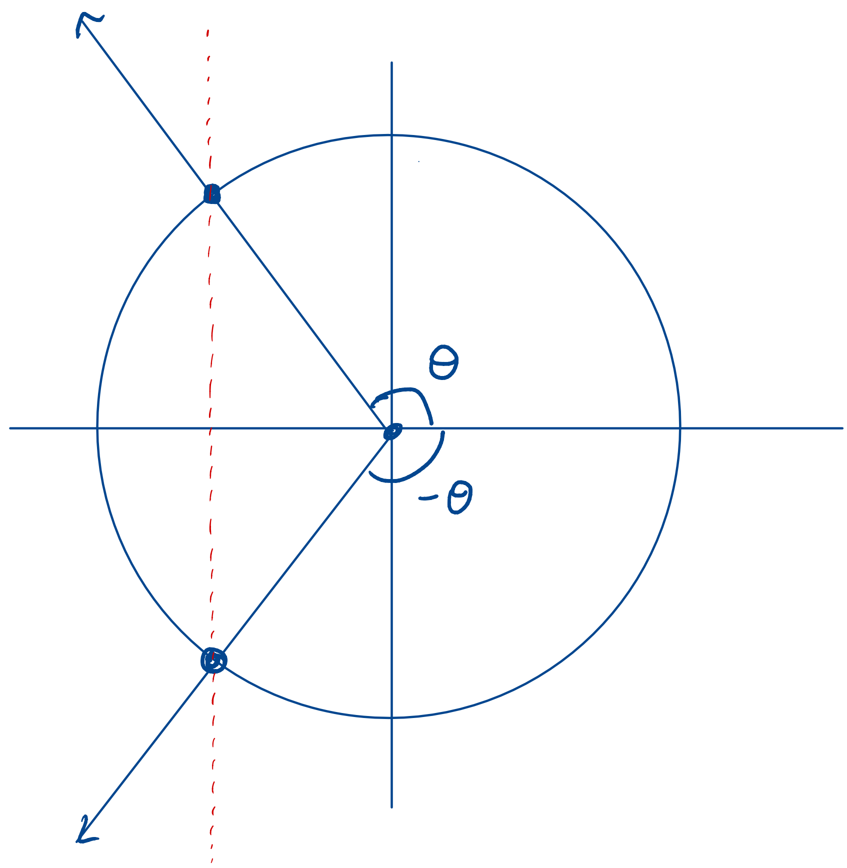
Q: Why is $\sin\theta$ odd?
Why is $\cos\theta$ even?

A: θ and $-\theta$ correspond to the same x-coord on the unit circle, but opposite y-coords

$\Downarrow$

$\cos(-\theta) = \cos(\theta)$

$\sin(-\theta) = -\sin(\theta)$



Simplifying Expressions

Example. Simplify $\dfrac{\sin\theta}{\tan\theta}$ to a form without quotients.

$$\frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \sin\theta \cdot \frac{\cos\theta}{\sin\theta} = \cos\theta$$

Verifying Identities

Example. Verify the identity:

$$\tan\theta + \cot\theta = \sec\theta\csc\theta.$$

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta}{\cos\theta \cdot \sin\theta} + \frac{\cos^2\theta}{\cos\theta \cdot \sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \cdot \sin\theta}$$

$$= \frac{1}{\cos\theta \cdot \sin\theta}$$

$$= \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}$$

$$= \sec\theta \cdot \csc\theta \quad \checkmark$$