The Other Trigonometric Functions

Previously, we focused on the trigonometric functions:

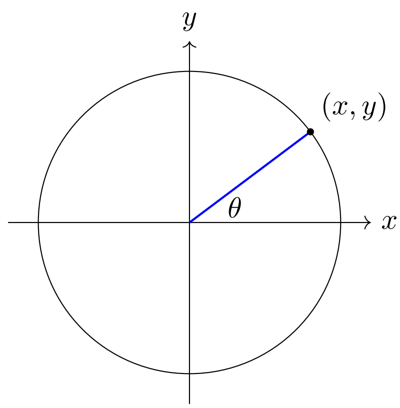
$$\sin\theta \quad \text{and} \quad \cos\theta.$$

Now we expand our study to the other four trigonometric functions:

$$\tan\theta, \quad \sec\theta, \quad \csc\theta, \quad \cot\theta.$$

Finding $\tan\theta$ from the Unit Circle

Recall that when an angle θ is drawn in standard position, the point where its terminal side intersects the unit circle has coordinates $(x, y) = (\cos\theta, \sin\theta)$.



The tangent of an angle θ is defined as

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

provided $x \neq 0$

Question. Explain why $\tan\theta$ can be interpreted as the slope of the terminal side of the angle θ.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y}{x} = \tan\theta$$

Question. What are the domain and range of $\tan\theta$?

vertical tangents (i.e. $x=0$)

Domain: Slopes exist everywhere except at angles coterminal with $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

$$\Rightarrow \quad \theta \neq \frac{\pi}{2} + k\pi \text{ where } k \text{ is an integer.}$$

Range: We can produce a terminal side[1] with any slope we want: $(-\infty, \infty)$

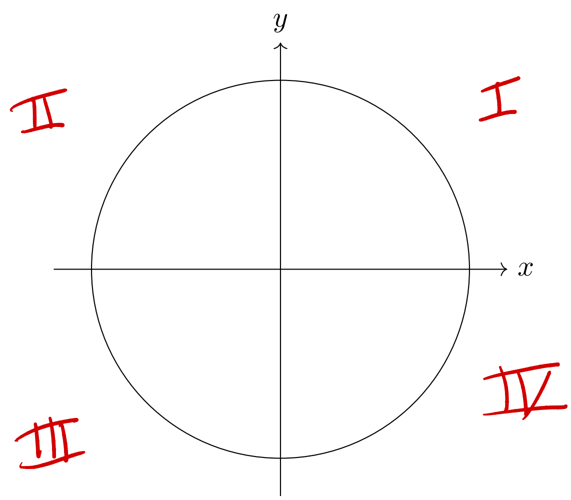
Finding $\sec\theta, \csc\theta$, and $\cot\theta$ from the Unit Circle

Definition. Let θ be an angle in standard position whose terminal side passes through the point (x, y) in the plane, and let $r = \sqrt{x^2 + y^2}$ be the distance from the point to the origin. Then the three additional trigonometric functions are defined by

$$\sec(\theta) = \frac{r}{x}, \quad \csc(\theta) = \frac{r}{y}, \quad \cot(\theta) = \frac{x}{y}.$$

When $r = 1$ (i.e. on the unit circle), these definitions reduce to

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$



	Function	Domain	Range
$x \neq 0 \rightarrow$	$\sec(\theta)$	$\theta \neq \frac{\pi}{2} + k\pi$	$(-\infty, -1] \cup [1, \infty)$
$y \neq 0 \rightarrow$	$\csc(\theta)$	$\theta \neq k\pi$	$(-\infty, -1] \cup [1, \infty)$
$y \neq 0 \rightarrow$	$\cot(\theta)$	$\theta \neq k\pi$	$(-\infty, \infty)$

k is an integer

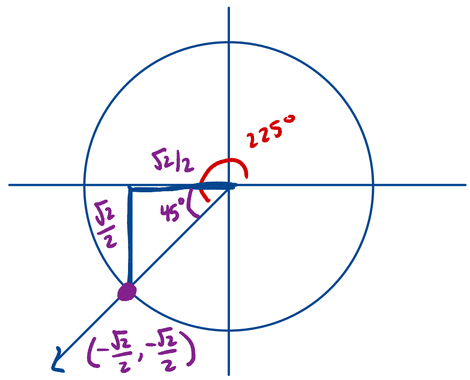
$-1 \leq \cos\theta \leq 1 \Rightarrow \sec\theta \geq 1$ or $\sec\theta \leq -1$

Function	I	II	III	IV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−
$\cot\theta$	+	−	+	−
$\sec\theta$	+	−	−	+
$\csc\theta$	+	+	−	−

slopes

$\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$, $\csc\theta = \frac{1}{y}$

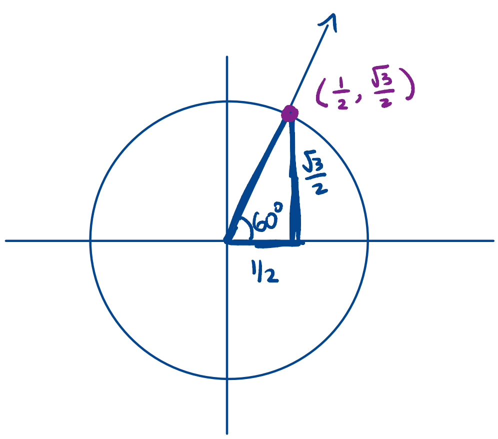
Example. Compute $\tan(225°)$.



$$\tan(225°) = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

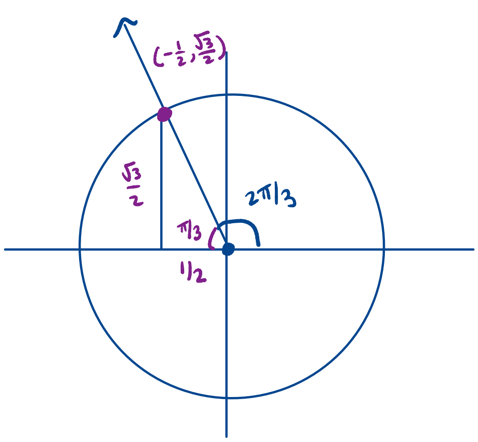
This is a line with slope 1.

Example. Compute $\sec\left(\frac{\pi}{3}\right)$.



$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos(\pi/3)} = \frac{1}{x} = \frac{1}{1/2} = 2$$

Example. Compute $\cot\left(\frac{2\pi}{3}\right)$.



$$\cot\left(\frac{2\pi}{3}\right) = \frac{\cos(2\pi/3)}{\sin(2\pi/3)} = \frac{x}{y} = \frac{-1/2}{\sqrt{3}/2}$$

$$= -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{-1}{\sqrt{3}} = \boxed{\frac{-\sqrt{3}}{3}}$$

$$\frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$