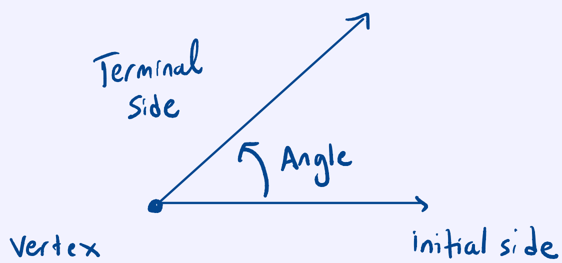
Angles, Radians, Area, and Arc Length

Angles as Measures of Rotation

Definition. An **angle** is defined as the measure of rotation between two rays (sides) sharing a common endpoint (vertex).

- **Initial Side**: the fixed, starting side.
- **Terminal Side**: the side that rotates about the vertex.



Definition. Angle measures can be positive or negative depending on the direction of rotation:

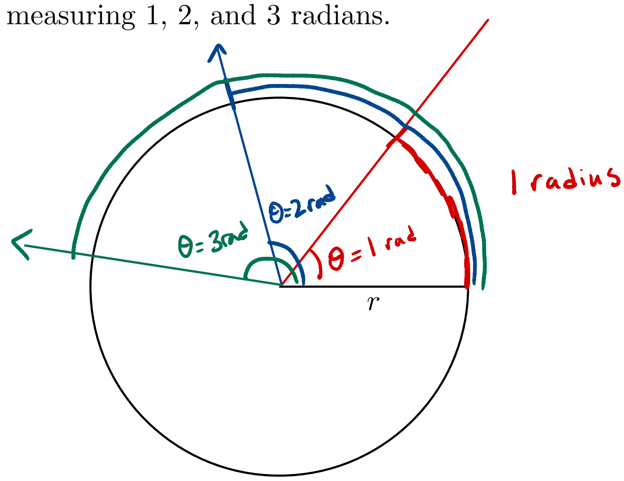
- Positive angles are defined by rotations in the counterclockwise direction.
- Negative angles are defined by rotations in the clockwise direction.

Units for Measuring Angles

Definition. Angles are measured in two primary units:

- Degrees: One complete rotation is 360°.
- Radians: A radian is defined as the angle traced out subtended at the center of a circle by an arc whose length is equal to the radius of the circle. One full rotation equals 2π radians.

Example. Draw rotations measuring 1, 2, and 3 radians.



Half of the circle is $\pi = 3.14$ rad

Since a full circle is 360° and also 2π radians, we derive the conversion factors:

$$1 \text{ radian} = \frac{180^\circ}{\pi}, \quad 1^\circ = \frac{\pi}{180} \text{ radians.}$$

$360^\circ = 2\pi$ radians

$1^\circ = \frac{2\pi}{360}$ radians $\qquad \left(\frac{360}{2\pi}\right)^\circ = 1$ rad

$1^\circ = \frac{\pi}{180}$ radians $\qquad \frac{180^\circ}{\pi} = 1$ radian

Example. Convert 45° to radians.

$$45^\circ = 45 \cdot 1^\circ = 45 \cdot \frac{\pi}{180} \text{ rad} = \frac{\pi}{4} \text{ rad}$$

Example. Convert $\frac{5\pi}{6}$ radians to degrees.

$$\frac{5\pi}{6} \text{ rad} = \frac{5\pi}{6} \cdot 1 \text{ rad} = \frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = 5 \cdot 30^\circ = 150^\circ$$

Example. Convert 120° to radians.

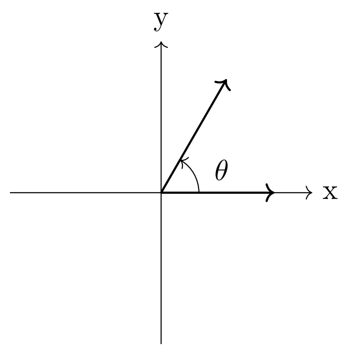
$$120 \cdot \frac{\pi}{180} = \frac{2\pi}{3} \text{ radians}$$

Example. Convert $\frac{7\pi}{4}$ radians to degrees.

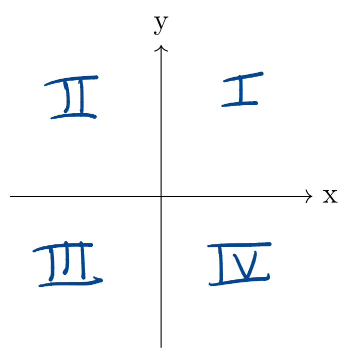
$$\frac{7\pi}{4} \text{ rad} = \frac{7\pi}{4} \cdot 1 \text{ rad} = \frac{7\pi}{4} \cdot \frac{180^\circ}{\pi} = 315^\circ$$

The Coordinate Plane

Definition. An angle is in **standard position** if its vertex is at the origin and its initial side lies along the positive x-axis.



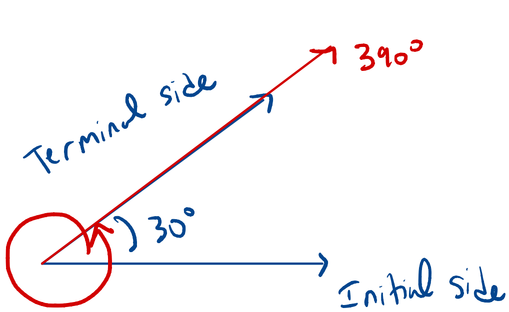
Definition. The Cartesian plane is divided into four quadrants:



Coterminal Angles

Definition. Coterminal angles share the same terminal side but differ by full rotations (360° or 2π rad).

Example. Find an angle coterminal with 30°.



Adding 360°:

$30° + 360° = 390°$

So 390° is coterminal with 30°

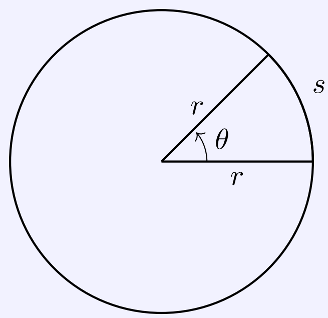
Arc Length

Definition. The **arc length** of a circle is the distance along the curved line forming the arc. The arc length s can be computed using:

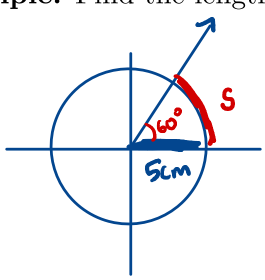
- **Radians:** $s = r\theta$ ← If θ is in radians, θ measures how many radii you have traced out. Multiplying by r gives the absolute distance.
- **Degrees:** $s = \dfrac{\pi r\theta}{180}$

where θ is the central angle in either radians or degrees, respectively.

Convert the angle to radians and use $s = r\theta$

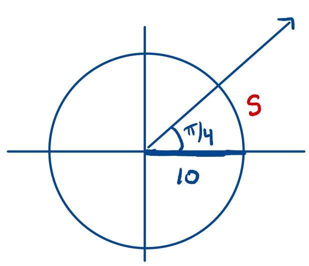


Example. Find the length of an arc in a circle with radius 5 cm and central angle 60°.



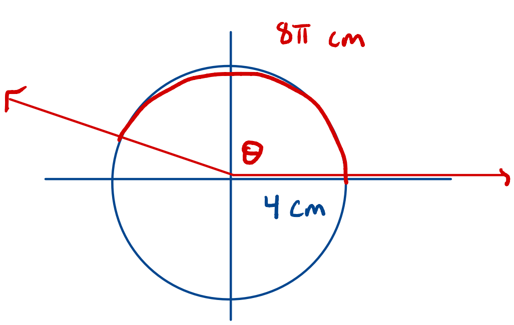
$$s = \frac{\pi}{180}\theta \cdot r = \frac{\pi}{180} \cdot 60 \cdot 5 = \frac{5\pi}{3} \text{ cm}$$

Example. Find the length of an arc in a circle with radius 10 m and central angle $\frac{\pi}{4}$ radians.



$$s = r \cdot \theta = 10 \cdot \frac{\pi}{4} = \frac{10\pi}{4} = \frac{5\pi}{2} \text{ m}$$

Example. A circle has an arc length of 8π cm and a radius of 4 cm. Find the central angle in radians.

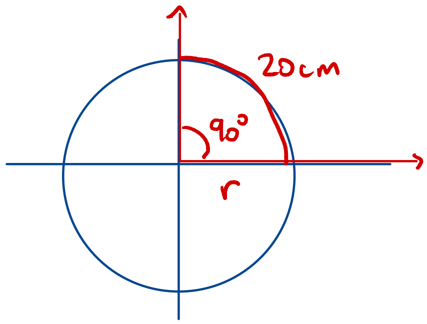


$$s = r \cdot \theta$$

$$8\pi = 4\theta$$

$$\theta = 2\pi$$

Example. A circle has an arc length of 20 cm and a central angle of 90°. Find the radius.



$$S = \frac{\pi}{180} r \theta$$

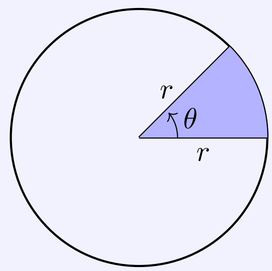
$$20 = \frac{\pi}{180} r \cdot 90^\circ$$

$$20 = \frac{\pi}{2} r \quad \Rightarrow \quad r = \frac{40}{\pi} \approx 12.73 \text{ cm}$$

Area of a Sector

Definition. The **sector area** A subtended by a central angle θ in a circle of radius r is given by:

$$A = \begin{cases} \frac{1}{2} r^2 \theta & \text{(radians)} \\ \frac{\pi r^2 \theta}{360} & \text{(degrees)} \end{cases}$$



Why? Area of circle is πr^2

In each case, decide what fraction of the circle the sector is...

① Degrees. The fraction is $\frac{\theta}{360}$.

$$A = \frac{\theta}{360} \cdot \pi r^2$$

② Radians. The fraction is $\frac{\theta}{2\pi}$.

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$

□

Example. Find the area of a sector in a circle with radius 6 cm and a central angle of 120°.

Example. Find the area of a sector in a circle with radius 10 m and a central angle of $\frac{\pi}{3}$ radians.

Example. A sector in a circle with radius 8 cm has an area of 32π cm^2. Find the central angle in radians.

Example. A sector has an area of 25π cm^2 and a central angle of 90°. Find the radius of the circle.