Solving Logarithmic Equations

Logarithmic equations play a crucial role in various real-world applications ranging from science and engineering to finance. They allow us to solve problems involving exponential growth and decay, measure sound intensity, calculate pH values, and even determine the magnitude of earthquakes.

Solving Logarithmic and Exponential Equations

When dealing with logarithmic equations, using the properties of logarithms can simplify the process. Consider the following examples.

Example.

$$\log_3(x) + \log_3(x-2) = 1$$

$\log_3(x(x-2)) = 1$ [Product property]

$3^1 = x(x-2)$ [Def. of log]

$x^2 - 2x - 3 = 0$ [Expand / rearrange]

$(x-3)(x+1) = 0$ [Factor]

$x = 3$ or $\cancel{x = -1}$ [Solve]

Since the domain of log is $x > 0$, the only valid solution is $x = 3$.

Remark. An *extraneous solution* is a solution obtained during the algebraic process that does not satisfy the original equation. Such solutions often appear when both sides of an equation are manipulated in a way that potentially introduces invalid answers.

Example.

$$\log(x-1)+\log(x+1)=\log(6)$$

$\log\big((x-1)(x+1)\big) = \log(6)$ [Product property]

$(x-1)(x+1) = 6$ [Equate the arguments]

$x^2-1 = 6$ [Solve]

$x^2 = 7$

$x = \pm\sqrt{7}$

For $\log(x-1)$, we need $x-1>0 \Rightarrow x>1$

For $\log(x+1)$, we need $x+1>0 \Rightarrow x>-1$

$\boxed{\text{only } \sqrt{7} \text{ is valid}}$

Example.

$$e^{2x}=5$$

$\ln(e^{2x}) = \ln(5)$ [Take log of both sides]

$2x\cdot\ln(e) = \ln(5)$ [Power property]

$2x\cdot 1 = \ln(5)$

$x = \dfrac{\ln(5)}{2}$ [Solve for x]

Using u-Substitution

Sometimes logarithmic equations exhibit a quadratic form. A common method to solve such equations is to use a substitution.

Example.

$$\ln^2 x - \ln x - 6 = 0.$$

Let $u = \ln(x)$. The equation becomes

$$u^2 - u - 6 = 0$$ [Use u-substitution]

$$(u-3)(u+2) = 0$$ [Factor]

$$u = 3 \quad \text{or} \quad u = -2$$

Substitute back:

$$\ln(x) = 3 \quad \text{or} \quad \ln(x) = -2$$

$$e^3 = x \quad \text{or} \quad e^{-2} = x$$ [Solve for x]

Modeling with Exponential and Logarithmic Functions

Exponential Growth and Decay

Exponential functions are widely used in modeling growth and decay processes. Two common forms are:

$$Ca^x \quad \text{and} \quad Ce^{kx},$$

where:

- C is a constant representing the initial value.
- a is the base in discrete growth/decay models.
- e^{kx} represents continuous growth/decay, where k is the rate constant.

Example. A bacteria culture grows exponentially according to the model

$$P(t) = P_0 e^{kt},$$

where P_0 is the initial population and k is the growth constant. Suppose the initial population is 100 bacteria and it doubles every 3 hours. Determine the time required for the population to reach 800 bacteria.

We want to find t such that

$$P(t) = 800$$

$$100e^{kt} = 800$$

Need to find k

Find k: Since the population doubles every 3 hours...

$$P(3) = 2P_0$$

$$100 \cdot e^{3k} = 200$$

$$e^{3k} = 2$$

$$\ln(e^{3k}) = \ln(2)$$

$$3k \cdot \ln(e) = \ln(2)$$

$$3k = \ln(2)$$

$$k = \frac{\ln(2)}{3}$$

$$100\, e^{\ln(2)/3 \cdot t} = 800$$

$$e^{\ln(2)/3 \cdot t} = 8$$

$$e^{\ln(2) \cdot t/3} = 8$$

$$\left(e^{\ln(2)}\right)^{t/3} = 8$$

$e^{\ln(2)} = 2$

$$2^{t/3} = 8$$

$$\log_2(2^{t/3}) = \log_2(8)$$

$$\frac{t}{3} \cdot \log_2(2) = 3 \Rightarrow t/3 = 3 \Rightarrow \boxed{t = 9 \text{ hours}}$$

Logarithmic Models in Applications

Logarithmic scales are used in various fields to compress wide-ranging values:

- **pH Scale:** Measures the acidity or alkalinity of a solution. It is defined as $\text{pH} = -\log[H^+]$.
- **Richter Scale:** Quantifies the magnitude of earthquakes on a logarithmic scale.
- **Sound Intensity:** Measured in decibels (dB), which is a logarithmic unit.

Example. The pH of a solution is defined by the formula

$$\text{pH} = -\log[H^+],$$

where $[H^+]$ is the concentration of hydrogen ions in moles per liter (M). If the pH of a solution is 3.2, find the value of $[H^+]$.

$$3.2 = -\log([H^+])$$

$$-3.2 = \log([H^+])$$

$$[H^+] = 10^{-3.2} \approx 6.31 \times 10^{-4}$$

Example. The Richter scale is defined by the equation:

$$M = \log\left(\frac{I}{I_0}\right),$$

where I is the intensity of the earthquake and I_0 is a reference intensity. Suppose an earthquake registers a magnitude of 5.5 on the Richter scale. Another earthquake is observed to be 500 times as intense as the first. Determine the magnitude of the stronger earthquake.

Let's suppose the first earthquake has intensity I_1:

$$5.5 = \log\left(\frac{I_1}{I_0}\right) \Rightarrow 10^{5.5} = \frac{I_1}{I_0} \Rightarrow I_1 = 10^{5.5} \cdot I_0$$

Since the second earthquake is 500 times as strong...

$$I_2 = 500 \cdot I_1 = 500 \cdot 10^{5.5} \cdot I_0$$

Hence

$$M_2 = \log\left(\frac{I_2}{I_0}\right) = \log\left(\frac{500 \cdot 10^{5.5} \cdot I_0}{I_0}\right) = \log\left(500 \cdot 10^{5.5}\right)$$

$$\approx 8.2$$

Example. The sound intensity level L in decibels (dB) is given by

$$L = 10 \log \left(\frac{I}{I_0} \right),$$

where I is the sound intensity and I_0 is a reference intensity. Suppose you hear a sound that measures 70 dB. How many times louder is this sound compared to the reference sound?

$$70 = 10 \log \left(\frac{I}{I_0} \right)$$

$$7 = \log \left(\frac{I}{I_0} \right)$$

$$10^7 = \frac{I}{I_0}$$

$$I = 10^7 \cdot I_0$$

Thus, the sound is 10,000,000 times as loud as the reference sond