Properties of Logarithms

Product Rule:

$$\log_b(A \cdot B) = \log_b(A) + \log_b(B) \longrightarrow X^A \cdot X^B = X^{A+B}$$

Quotient Rule:

$$\log_b\!\left(\frac{A}{B}\right) = \log_b(A) - \log_b(B) \longrightarrow \frac{x^A}{x^B} = x^{A-B}$$

Power Rule:

$$\log_b(A^c) = c\,\log_b(A) \longrightarrow (X^A)^c = X^{c\cdot A}$$

Example. Simplify $\log(5^3)$.

$$\log(5^3) = 3 \cdot \log(5)$$

Example. Evaluate $\log_2(8^3)$.

$$\log_2(8^3) = 3\log_2(8) = 3 \cdot 3 = 9$$

Example. Evaluate $\log_3\!\left(\frac{81}{9}\right)$.

$$= \log_3(9) = 2$$

OR

$$= \log_3(81) - \log_3(9) = 4 - 2 = 2$$

Example. Expand $\log\left(\frac{x^2}{y^3}\right)$.

$$\log\left(\frac{x^2}{y^3}\right) = \log(x^2) - \log(y^3)$$

$$= 2\log(x) - 3\log(y)$$

Example. Combine $\log_3(15) - \log_3(40) + \log_3(72)$ into a single logarithm and simplify.

$$= \log_3\left(\frac{15}{40}\right) + \log_3(72)$$

$$= \log_3\left(\frac{15 \cdot 72}{40}\right)$$

$$= \log_3(27)$$

$$= 3$$

Change of Base Formula: For any positive A and bases $b, c > 0$ (with $b \neq 1$ and $c \neq 1$),

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$$

Example. Simplify $\log_5(14)$ using base 10 and base e.

$$\log_5(14) = \frac{\log(14)}{\log(5)}$$

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Example. Evaluate $\log_2(64^3)$ without directly computing 64^3.

$$\begin{aligned} \log_2(64^3) &= 3\log_2(64) \\ &= 3\log_2(8^2) \\ &= 6\log_2(8) \\ &= 6 \cdot 3 \\ &= 18 \end{aligned}$$

OR

$$\begin{aligned} &= \log_2\left((2^6)^3\right) \\ &= \log_2(2^{18}) \\ &= 18\log_2(2) \\ &= 18 \cdot 1 \\ &= 18 \end{aligned}$$

Example. Expand $\log\left(\sqrt{x^2+2}\right)$.

$$\log\left(\sqrt{x^2+2}\right) = \log\left((x^2+2)^{1/2}\right) = \frac{1}{2}\log(x^2+2)$$

Example. Expand the logarithm:

$$\log\Big(\frac{x^3\sqrt{y}}{(z^2)^3}\Big).$$

$$\log\Big(\frac{x^3y^{1/2}}{z^6}\Big) = \log(x^3y^{1/2}) - \log(z^6)$$
$$= \log(x^3) + \log(y^{1/2}) - \log(z^6)$$
$$= 3\log(x) + \tfrac{1}{2}\log(y) - 6\log(z)$$

Example. Combine and simplify the following expression into a single logarithm:

$$2\ \log(x)\ -\ \tfrac{1}{2}\ \log(y)\ +\ \log(5).$$

$$\log(x^2) - \log(y^{1/2}) + \log(5) = \log\Big(\frac{x^2}{y^{1/2}}\Big) + \log(5)$$
$$= \log\Big(\frac{x^2}{y^{1/2}}\cdot 5\Big)$$
$$= \log\Big(\frac{5x^2}{\sqrt{y}}\Big)$$

Example. Evaluate the expression:

$$\log_2\Big(\frac{16^3}{8^2}\Big).$$

$$\log_2(16^3) - \log_2(8^2)$$

$$3\log_2(16) - 2\log_2(8)$$

$$3\cdot 4 - 2\cdot 3$$

$$12 - 6$$

$$6$$